CHAPTER 1 Basic Concepts and Definitions of Graph Theory

1.1 INTRODUCTION

Graph theory is a branch of mathematics started by Euler [45] as early as 1736. It took a hundred years before the second important contribution of Kirchhoff [139] had been made for the analysis of electrical networks. Cayley [22] and Sylvester [228] discovered several properties of special types of graphs known as *trees*. Poincaré [195] defined in principle what is known nowadays as the incidence matrix of a graph. It took another century before the first book was published by König [141]. After the second world war, further books appeared on graph theory, Ore [183], Behzad and Chartrand [11], Tutte [240], Berge [13], Harary [70], Gould [63], and West [245], among many others.

Graph Theory has found many applications in engineering and science, such as chemical, civil, electrical and mechanical engineering, architecture, management and control, communication, operational research, sparse matrix technology, combinatorial optimisation, and computer science. Therefore many books have been published on applied graph theory such as those by Bondy and Murty [16], Chen [24], Thulasiraman and Swamy [235], Wilson and Beineke [252], Mayeda [170], Christofides [26], Gondran and Minoux [61], Beineke and Wilson [12], Deo [37], Cooke et al. [32], Kaveh [110-111] and many others. In recent years, due to the extension of the concepts and applications of the graph theory, many journals such as *Journal of Graph Theory, Journal of Combinatorial Theory A & B, Discrete and Applied Mathematics, SIAM Journal of Discrete Mathematics, European Journal of Combinatorics*, and Graphs and Combinatorics are being published to cover the advances made in this field.

In this chapter basic definitions and concepts of graph theory are presented; however, for proofs and details the reader may refer to textbooks on this subject, Refs [63,70,245].

1.2 BASIC DEFINITIONS

There are many physical systems whose performance depends not only on the characteristics of their components, but also on their relative location. As an example, in a structure, if the properties of a member are altered, the overall behaviour of the structure will be changed. This indicates that the performance of a structure depends on the characteristics of its members. On the other hand, if the location of a member is changed, the properties of the structure will again be different. Therefore the connectivity (topology) of the structure influences the performance of the whole structure. Hence it is important to represent a system so that its topology can be understood clearly. The graph model of a system provides a powerful means for this purpose.

1.2.1 DEFINITION OF A GRAPH

A graph S consists of a set N(S) of elements called *nodes* (vertices or points) and a set M(S) of elements called *members* (edges or arcs) together with a relation of *incidence* which associates each member with a pair of nodes, called its *ends*.

Two or more members joining the same pair of is are known as a *multiple member*, and a member joining a node to itself is called a *loop*. A graph with no loops and multiple members is called a *simple graph*. If N(S) and M(S) are countable sets, then the corresponding graph S is *finite*. Since the great majority of the results in this book pertain to finite graphs with no loop and multiple members, only simple finite graphs are needed, which are referred to as *graphs*.

The above definitions correspond to *abstract graphs*; however, a graph may be visualized as a set of points connected by line segments in Euclidean space; the points are identified with nodes, and the line segments without their end points are identified with members. Such a configuration is known as a *topological graph*. These definitions are illustrated in Figure 1.1.



(a) A non-simple graph.

(b) A simple graph.

Fig. 1.1 Non-simple and simple graphs.

1.2.2 ADJACENCY AND INCIDENCE

Two nodes of a graph are called *adjacent* if these nodes are the end nodes of a member. A member is called *incident with a node* if it is an end node of that member. Two members are called *incident* if they have a common end node. The *degree* (valency) of a node n_i of a graph, denoted by deg (n_i) , is the number of members incident with that node. Since each member has two end nodes, the sum of node-degrees of a graph is twice the number of its members (handshaking lemma - known as the first theorem of graph theory).



Fig. 1.2 A simple graph S.

As an example, in Figure 1.2 two nodes n_4 and n_5 are adjacent. Node n_3 is incident with member m_2 and m_6 , and deg $(n_2) = 4$.

1.2.3 ISOMORPHIC GRAPHS

Two graphs S_1 and S_2 are called *isomorphic* if there exists a one-to-one correspondence between their node sets and adjacency is preserved. As an example, the three graphs shown in Figure 1.3 are isomorphic. The word isomorphic is derived from the Greek words *same* and *form*.





3

1.2.4 GRAPH OPERATIONS

A subgraph S_i of S is a graph for which $N(S_i) \subseteq N(S)$ and $M(S_i) \subseteq M(S)$, and each member of S_i has the same ends as in S.

The union of subgraphs $S_1, S_2, ..., S_k$ of S, denoted by $S^k = \bigcup_{i=1}^k S_i = S_1 \cup S_2 \cup ...$ $\cup S_k$, is a subgraph of S with $N(S^k) = \bigcup_{i=1}^k N(S_i)$ and $M(S^k) = \bigcup_{i=1}^k M(S_i)$. The *intersection* of two subgraphs S_i and S_j is similarly defined using intersections of node-sets and member-sets of the two subgraphs. The *ring sum* of two subgraphs $S_i \oplus S_j = S_i \cup S_j - S_i \cap S_j$ is a subgraph which contains the nodes and members of S_i and S_j except those elements common to S_i and S_j . These definitions are illustrated in Figure 1.4.



Fig. 1.4 A graph, two of its subgraphs, union, intersection and ring sum.

There are many other useful operations, such as Cartesian product, direct product and strong Cartesian product, successfully applied to structural engineering [129].

1.2.5 WALKS, TRAILS AND PATHS

A walk w of S, is a finite sequence $w = \{n_0, m_1, n_1, ..., m_k, n_k\}$ whose terms are alternately nodes n_i and members m_i of S for $1 \le i \le k$, and n_{i-1} and n_i are the two ends of m_i . A *trail* t in S, is a walk in which no member of S appears more than once. A *path* P in S, is a trail in which no node appears more than once. The *length* of a path P_i denoted by $L(P_i)$ is taken as the number of its members. P_i is called the *shortest path* between the two nodes n_0 and n_k , if for any other path P_i

between these nodes $L(P_i) \le L(P_j)$. The *distance* between two nodes of a graph is defined as the number of its members of a shortest path between these nodes.

As an example, in Figure 1.5,

 $w = (n_1, m_5, n_5, m_6, n_2, m_7, n_6, m_{11}, n_5, m_6, n_2, m_2, n_3)$

is a walk between n_1 and n_3 , in which member m_6 and nodes n_2 and n_5 are repeated twice.



.

is a trail between n_1 and n_4 in which node n_5 is repeated twice.

 $P = (n_1, m_5, n_5, m_6, n_2, m_7, n_6)$

is a path of length 3 in which no node and no member is repeated.

1.2.6 CONNECTEDNESS

Two nodes n_i and n_j are said to be *connected* in S if there exists a path between these nodes. A graph S is called *connected* if all pairs of its nodes are connected. A *component* of a graph S is a maximal connected subgraph, i.e. it is not a subgraph of any other connected subgraph of S. These definitions are illustrated in Figure 1.6.



(a) A connected graph.

(b) A disconnected graph.

Fig. 1.6 A connected graph and a disconnected graph with 3 components.

1.2.7 CYCLES AND CUTSETS

A cycle is a path $(n_0, m_1, n_1, ..., m_p, n_p)$ for which $n_0 = n_p$ and $p \ge 1$; i.e. a cycle is a closed path. Similarly, a *closed trail (hinged cycle)* and a *closed walk* can be defined, Figure 1.7.



A *cutset* in a graph S is a set of members whose removal from the graph increases the number of connected components of S, Figure 1.8(a). If a cutset results in two components S_1 and S_2 , then it is known as *prime cutset*, Figure 1.8(b). A *link* is a member with its ends in two components produced by a cutset. Links are shown in bold lines. In a prime cutset, if one of the components S_1 or S_2 consists of a single node, then the prime cutset is called a *cocycle*, Figure 1.8(c).



Fig. 1.8 Different cutsets of S.

1.2.8 TREES, SPANNING TREES AND SHORTEST ROUTE TREES

A *tree* T of S is a connected subgraph of S, which contains no cycle. A set of trees of S forms a *forest*. If a tree contains all the nodes of S, it is called a *spanning tree* of S. For simplicity it will be referred to as a *tree*, from now on.

A *shortest route tree* (SRT) rooted at a specified node n_0 of S, is a tree for which the distance between every node n_j of T and n_0 is a minimum. An SRT of a graph can be generated by the following simple algorithm:

Label the selected root as "0" and the adjacent nodes as "1". Record the members incident to "0" as tree members. Repeat the process of labelling with "2" the unnumbered ends of all the members incident with nodes labelled as "1", again

recording the tree members. This process terminates when each node of S is labelled and all the tree members are recorded. The label of the last node indicates the *length* of the SRT and the maximum number of nodes with the same label is defined as the *width* of the SRT.

The above definitions are illustrated in Figure 1.9. The length and width of the SRT in Figure 9(d) are 2 and 3, respectively.

It is easy to prove that for a tree T,

$$M(T) = N(T) - 1, (1-1)$$

where M(T) and N(T) are the numbers of members and nodes of T, respectively.

The complement of T in S is called a *cotree*, denoted by T*. The members of T are known as *branches* and those of T* are called *chords*. For a connected graph S, the number of chords is given by:

$$M(T^*) = M(S) - M(T).$$
 (1-2)

Since, N(T) = N(S), hence,

$$M(T^*) = M(S) - N(S) + 1, \qquad (1-3)$$

where M(S) and N(S) are the numbers of members and nodes of S, respectively. Notice that for a set and its cardinality the same notation is used and the difference should be obvious from the context.



(d) An SRT rooted from n_0 . (e) A cotree shown in dashed lines. Fig. 1.9 A tree and a cotree of S.

1.3 DIFFERENT TYPES OF GRAPHS

In order to simplify the study of properties of graphs, different types of graphs have been defined. Some important ones are as follows:

A *null graph* is a graph which contains no members. Thus N_k is a graph containing k isolated nodes.

A cycle graph is a graph consisting of a single cycle. Therefore C_k is a polygon with k members.

A *path graph* is a graph consisting of a single path. Hence P_k is a path with k nodes and (k-1) members.

A *complete graph* is a graph in which every two distinct nodes are connected by exactly one member, Figure 1.10.



A complete graph with N nodes is denoted by K_N . It is easy to prove that a complete graph with N nodes has N(N - 1)/2 members.

A graph is called *bipartite*, if the corresponding node set can be split into two sets N_1 and N_2 in such a way that each member of S joins a node of N_1 to a node of N_2 . A *complete bipartite* graph is a bipartite graph in which each node N_1 is joined to each node of N_2 by exactly one member. If the number of nodes in N_1 and N_2 are denoted by r and s, respectively, then a complete bipartite graph is denoted by $K_{r,s}$. Examples of bipartite and complete bipartite graphs are shown in Figure 1.11.







(b) A complete bipartite graph $K_{3,4}$



1.4 VECTOR SPACES ASSOCIATED WITH GRAPHS

In this section, it is shown that a vector space can be associated with a graph, and the properties of two important subspaces of this vector space, namely cycle and cutset spaces, is studied. For this purpose, simple definitions from sets, groups, fields and vector spaces are briefly presented. The material presented in this section is based on the work of Thulasiraman and Swamy [235].

1.4.1 GROUPS AND FIELDS

Consider a finite set $S = \{a, b, c, ...\}$, and define a binary operation + on S. This operation assigns to every pair (a and b) \in S a unique element denoted by a+b. The set S is said to be *closed under* + if the element (a+b) \in S, whenever (a and b) \in S.

The operation + is said to be *associative* if a+(b+c) = (a+b)+c for all a, b and c in S. The operation is called *commutative* if a+b = b+a for all a and b in S.

Definition 1: A set S with a binary operation +, called addition, is a *group* if the following postulates hold:

1. S is closed under +.

2. The operation + is associative.

3. There exists a unique element $e \in S$ such that a+e = e+a = a for all $a \in S$.

4. For each element $a \in S$ there exits a unique element b such that b+a = a+b = e. The element b is known as the *inverse* of a, and vice versa. Obviously the identity element e is its own inverse.

A group is called *abelian* if the operation + is commutative.

Examples: The set $S = \{..., -2, -1, 0, +1, +2, ...\}$ consisting of all integer numbers under usual addition for + forms a group. Here, 0 is the identity element and –a is the inverse of $a \in S$. This group is also abelian.

Another example is the set $Z_p = \{0, 1, 2, \dots, p-1\}$ of integers with modulo p addition operation. If a = mp+q for $0 \le q \le p-a$, then in modulus arithmetic a = q(modulo p). In this group, 0 is the identity element and the integer p-a is the inverse of a, except 0 which is its own inverse. As an example, the addition table of Z_3 is shown in Table 1.1.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

	able1.1	1 Table of	f Z3.
--	---------	------------	-------

Definition 2: A set F with two operations of addition (+) and multiplication (\circ) is a *field* if the following postulates hold:

1. F is abelian group under +, with the identity element denoted as e.

2. The set $F - \{e\}$ is an abelian group under \circ , the multiplication operation.

3. The multiplication operation is distributive with respect to addition, i.e.

 $a \circ (b+c) = (a \circ b) + (a \circ c)$ for all a, b and c in F.

Example: Consider $Z_p=\{0,1,2, \ldots, p-1\}$ again with addition (modulo p) and multiplication (modulo p) as the two operation. It can be shown that the set $Z_p-\{0\}=\{1,2, \ldots, p-1\}$ is an abelian group if p is prime. Therefore Z_p is a field if p is a prime. The set Z_2 of integers modulo 2, denoted GF(2), is an important field in our study with:

0+0 = 0, 1+0 = 0+1 = 1, and 1+1=0, $0 \circ 0 = 0, 1 \circ 0 = 0 \circ 1 = 0, \text{ and } 1 \circ 1 = 1.$

1.4.2 VECTOR SPACES

Consider a set S with a binary operation \Box . Let F be a field with + and \circ being the addition and multiplication operations, respectively. A multiplication operation, denoted by *, is also defined between the elements of F and those of S. This operation assigns to each ordered pair (α ,s) a unique element denoted by α *s, where α is in F and s is in S. The set S is a vector space over F if the following postulates hold:

1. S is an abelian group under \square .

2. For any elements α and β in F, and any elements s_1 and s_2 in S the followings hold:

$$\alpha * (\mathbf{s}_1 \bullet \mathbf{s}_2) = (\alpha * \mathbf{s}_1) \bullet (\alpha * \mathbf{s}_2)$$

and

$$(\alpha + \beta) * s_1 = (\alpha * s_1) \boxdot (\beta * s_1)$$

3. For any element α and β in F and any element s in S:

$$(\alpha * \beta) * s = \alpha * (\beta * s)$$

4. For any element s in S, 1*s = s, where 1 is the multiplicative identity in F.

Consider a vector space S, over the field F. The elements of S are called *vectors* and those of F are known as *scalars*. If an element s of S is expressible as,

$$\mathbf{s} = (\alpha_1 \ast \mathbf{s}_1) \boxdot (\alpha_2 \ast \mathbf{s}_2) \boxdot \dots \boxdot (\alpha_i \ast \mathbf{s}_i), \tag{1-4}$$

where s_i 's are vectors and α_i 's are scalars, then s is said to be a *linear combination* of s_1, s_2, \ldots, s_j . Vectors s_1, s_2, \ldots, s_j are said to be *linearly independent* if no vectors in this set is expressible as a linear combination of the remaining vectors in the set. Vectors s_1, s_2, \ldots, s_n form a *basis* in the vector space S if they are linearly independent and every vector in S is expressible as a linear combination of these vectors. The vectors s_1, s_2, \ldots, s_n are known as *basis vectors*. The *dimension* of the vector space S, denoted by dim(S), is the number of vectors in a basis of S. If S' is a subset of the vector space S over F, then S' is a subspace of S if S' is also a vector space on F. The direct sum $S_1 \boxdot S_2$ of two subspaces S_1 and S_2 of S is the set of all vectors of the form $s_1 \boxdot s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$. It can be proved that $S_1 \boxdot S_2$ is also a subspace, and its dimension is given by

$$\dim(S_1 \boxdot S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2).$$
(1-5)

Note that $S_1 \cap S_2$ is also a subspace whenever S_1 and S_2 are subspaces.

1.4.3 VECTOR SPACE OF A GRAPH

Consider a graph S = (N,M) and let W_S denote the collection of all subsets of M, including the empty set \emptyset . Under ring sum operation \oplus between sets, in the following it is shown that W_S is an abelian group. Defining a suitable multiplication between elements of the field Z_2 and those of W_S , it can be shown that W_S is a vector space over Z_2 .

It can be shown that W_S is closed under \oplus . The operator \oplus is associative and commutative. Further for any element M_i in W_S , $M_i \oplus \emptyset = M_i$ and $M_i \oplus M_i = \emptyset$. Therefore for the operation \oplus , \emptyset is the identity element, and each M_i is its own inverse. Hence W_S is an abelian group under \oplus .

Let *, a multiplication operation between the elements of Z_2 and those of W_S be defined as follows:

$$1 * M_i = M_i$$
 and $0 * M_i = \emptyset$.

With this definition of * one can verify that the elements of W_S satisfy the following other requirements of a vector space:

1. $(\alpha+\beta)*M_i = (\alpha*M_i) \oplus (\beta*M_i)$. 2. $\alpha*(M_i \oplus M_j) = (\alpha*M_i) \oplus (\alpha*M_j)$.

3. $(\alpha.\beta)*M_i = \alpha*(\beta*M_i)$.

4. $1*M_i = M_i$. (Notice that 1 is the multiplicative identity in Z₂.)

Therefore W_S is a vector space over Z_2 . The dimension of this space is equal to the number of members of the graph S.

Since each member-induced subgraph of S corresponds to a unique subset of M, and by definition the ring sum of any two member-induced subgraphs corresponds to the ring sum of their corresponding member sets, it is obvious that the set of all members-induced subgraphs of S is also a vector space over Z_2 if the multiplication is defined as follows:

 $1*M_i = M_i$ and $0*M_i = \emptyset$, the null graph having no nodes and no members.

This vector space will also be referred to by the symbol W_s.

1.4.4 CYCLE SUBSPACE AND CUTSET SUBSPACE OF A GRAPH

Now we study two important subspaces of W_S , namely cycle space and cutset space of a graph.

Theorem 1: The set of all simple cycles and union of member-disjoint cycles of a graph, W_C , is a subspace W_S of S.

Consider C_1 and C_2 as two cycles of W_C . To prove the theorem one should show that $C_1 \oplus C_2$ is also a cycle C_3 belonging to W_C . Let n be a node of C_3 . This node is present at least in of the two cycles C_1 and C_2 . Let M_i (i=1,2,3), denotes the members incident to n in C_i . Let $|C_i|$ shows the number of members of C_i , thus $|M_i|$ is the number of members incident to n in C_i . Note that $|M_1|$ and $|M_2|$ are both even and one of them may be zero. Furthermore $|M_3|$ is non-zero.

Since $C_3 = C_1 \oplus C_2$, we have:

$$M_3 = M_1 \oplus M_2$$

Therefore:

$$|\mathbf{M}_3| = |\mathbf{M}_1| \oplus |\mathbf{M}_2| - 2|\mathbf{M}_1 \cap \mathbf{M}_2|.$$

Since $|M_1|$ and $|M_2|$ are both even, thus $|M_3|$ is also even. This is true for all the nodes of C₃, and it follows that it is a cycle in W_C and the theorem is proven.

Theorem 2: The set of all cutsets and the union of member-disjoint cutsets $W_{C'}$ in a graph S, is a subspace of the vector space W_S of S.

It can be shown that the ring sum of any two cuts in a graph is a cut in S. Similarly the union of any two member-disjoint cuts in a graph S is also a cut in S. Since $W_{C'}$ is closed under the ring sum operation thus the proof is completed.

1.4.5 FUNDAMENTAL CYCLE BASES

A special cycle basis known as a *fundamental cycle basis* can easily be constructed corresponding to a tree T of S. In a connected S, a chord of T together with T contains a cycle known as a *fundamental cycle* of S. Moreover, the fundamental cycles obtained by adding the chords to T, one at a time, are independent, because each cycle has a member which is not in the others. Also, every cycle C_i depends on the set of fundamental cycles obtained by the above process, for C_i is the symmetric difference of the cycles determined by the chords of T which lie in C_i . Thus the cycle rank (cyclomatic number, first Betti number, nullity) of graph S which is the number of cycles in a basis of the cycle space of S, is given by,

$$b_1(S) = M(S) - N(S) + 1,$$
 (1-6)

and if S contains $b_0(S)$ components, then:

$$b_1(S) = M(S) - N(S) + b_0(S).$$
 (1-7)

A formal proof is provided in Section 1.4.7.

As an example, the selected tree T and four fundamental cycles of S are illustrated in Figure 1.12.



Fig. 1.12 A graph S and a fundamental cycle basis of S.

1.4.6 FUNDAMENTAL CUTSET BASES

A basis can be constructed for the cutset space of a graph S. Consider the tree T and its cotree T*. The subgraph of S consisting of T* and any member of T (branch) contains exactly one cutset known as a *fundamental cutset*. The set of cutsets obtained by adding branches of T to T*, one at a time, forms a basis for the cutset space of S, known as a *fundamental cutset basis* of S. The *cutset rank* (rank of S) is the number of cutsets in a basis for the cutset space of S, which is given by

$$\rho(S) = N(S) - 1,$$
 (1-8)

and for a graph with $b_0(S)$ components:

$$\rho(S) = N(S) - b_0(S).$$
(1-9)

A formal proof is provided in Section 1.4.7.

A graph S and a fundamental cutset basis of S are shown in Figure 1.13. A branch of the tree subdivides the nodes of the tree into two subsets. The members of a cutest should have one end in each subsets.







Fig. 1.13 A graph S and a fundamental cutset basis of S.

1.4.7 DIMENSION OF CYCLE AND CUTSET SUBSPACES

Consider T as a spanning tree of a connected graph S with N nodes and M members. The branches of T are denoted by $b_1, b_2, \ldots, b_{N-1}$ and the chords by $c_1, c_2, \ldots, c_{M-N+1}$. Let C_i and C'_i refer to the fundamental cycle and the fundamental cutset with respect to c_i and b_i , respectively.

Since each fundamental cycle contains exactly one chord, and this chord is present in no other fundamental cycle, therefore the fundamental cycles $C_1, C_2, \ldots, C_{M-N+1}$ are independent. Using a similar reasoning for cutsets, it becomes obvious that all the fundamental cutsets $C'_1, C'_2, \ldots, C'_{N-1}$ are also independent.

Now we need to prove that every subgraph in cycle (cutset) subspace of S can be expressed as a ring sum of C_i (C'_i). For this purpose consider any subgraph C in the cycle space of S. Let C contain the chords $c_{i1}, c_{i2}, ..., c_{ir}$. Let C' denote the ring sum of the fundamental cycle $C_{i1}, C_{i2}, ..., C_{ir}$. Obviously the chords $c_{i1}, c_{i2}, ..., c_{ir}$ are present in C', and C' contains no other chord of the T. Since C also contains these chords and no others, C' \oplus C contains no chords.

Now it is claimed that $C' \oplus C$ is empty. If this is not true, then by the preceding arguments, $C' \oplus C$ contains only branches and has no cycle. On the other hand, being a ring sum of cycles, $C' \oplus C$ is a cycle of the union of member-disjoint cycles. Therefore the assumption that $C' \oplus C$ is not empty leads to a contradiction. Hence $C' \oplus C$ is empty. This implies that $C = C' = C_{i1} \oplus C_{i2} \oplus \ldots \oplus C_{ir}$, i.e. every subgraph in the cycle space of S can be expressed as a ring sum of the fundamental cycles.

In a similar manner it can be proved that every subgraph in the cutset subspace of S can be expressed as a ring sum of the fundamental cutsets.

The following fact can now be concluded:

1. The fundamental cycles with respect to a spanning tree of S constitute a basis for the cycle subspace of S, and therefore the dimension of the cycle subspace of S is equal to M-N+1.

2. The fundamental cutsets with respect to a spanning tree of S constitute a basis for the cutset subspace of S, and therefore the dimension of the cutset subspace of S is equal to N-1.

For graphs which are not connected, spanning forest will replace the spanning tree, and the dimensions for cycle subspace and cutset subspace will then be nullity of S = $b_1(S) = M-N+b_0(S)$ and rank of S = $\rho(S) = N-b_0(S)$, respectively. Here, $b_0(S)$ is the number of components of S.

1.4.8 ORTHOGONALITY PROPERTY

Two vectors are called *orthogonal* if their scalar product is zero. It can be shown that a vector is a cycle set (cutset) vector, if and only if it is orthogonal to every vector of a cutset (cycle set) basis. Since the cycle set and cutset spaces of a graph S containing M(S) members are both subspaces of the M(S)-dimensional space of all vectors which represent subsets of the members, therefore the cycle set and cutset spaces are *orthogonal components* of each other.

1.5 MATRICES ASSOCIATED WITH A GRAPH

Matrices play a dominant role in the theory of graphs and in particular in its applications to structural analysis. Some of these matrices conveniently describe the connectivity properties of a graph and others provide useful information about the patterns of the structural matrices, and some reveal additional information about transformations such as those of equilibrium and compatibility equations.

In this section various matrices are studied which reflect the properties of the corresponding graphs. For simplicity, all the graphs are assumed to be connected, since the generalization to non-connected graphs is trivial and consists of considering the direct sum of the matrices for their components.

1.5.1 MATRIX REPRESENTATION OF A GRAPH

A graph can be represented in various forms. Some of these representations are of theoretical importance, others are useful from the programming point of view when applied to realistic problems. In this section six different representations of a graph are described.

Node Adjacency Matrix: Let S be a graph with N nodes. The *adjacency matrix* **A** is an N×N matrix in which the entry in row i and column j is 1 if node n_i is adjacent to n_j , and is 0 otherwise. This matrix is symmetric and the row sums of **A** are the degrees of the nodes of S.

The adjacency matrix of the graph S, shown in Figure 1.14, is a 5×5 matrix as:



Fig. 1.14 A graph S.

It can be noted that **A** is a symmetric matrix of trace zero. For two isomorphic graphs S and S', the adjacency matrix **A** of S can be transformed to **A**' of S' by simultaneous permutations of the rows of **A**. The (i,j)th entry of A^2 shows the number of walks of length 2 with n_i and n_j as end nodes. Similarly, the entry in the

(i,j) position of $\mathbf{A}^k\,$ is equal to the number of walks of length k with n_i and n_j as end nodes. The polynomial,

$$\phi(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}), \qquad (1-11)$$

is called the *characteristic polynomial* of S. The collection of N(S) eigenvalues of A is known as the *spectrum* of S. Since A is symmetric, the spectrum of S consists of N(S) real numbers. The sum of eigenvalues of A is equal to zero. The eigenvectors of A are orthogonal.

Node-Member Incidence Matrix: Let S be a graph with M members and N nodes. The *node-member incidence matrix* $\overline{\mathbf{B}}$ is an N×M matrix in which the entry in row i and column j is 1 if node n_i is incident with member m_j, and is 0 otherwise. As an example, the node-member incidence matrix of the graph in Figure 1.14 is a 5×7 matrix of the form:

Obviously, the pattern of an incidence matrix depends on the particular way that its nodes and members are labelled. One incidence matrix can be obtained from another by simply interchanging rows (corresponding to relabelling the nodes) and columns (corresponding to relabelling the members).

The incidence matrix $\overline{\mathbf{B}}$ and the adjacency matrix \mathbf{A} of a graph S are related by,

$$\overline{\mathbf{B}}\overline{\mathbf{B}}^{\mathrm{t}} = \mathbf{A} + \mathbf{V},\tag{1-13}$$

where **V** is a diagonal matrix of order N(S), known as the *degree matrix*, whose typical non-zero entry v_{ii} is the valency of the node n_i of S for i=1, ..., N(S).

The rows of $\overline{\mathbf{B}}$ are dependent and one row can arbitrarily be deleted to ensure the independence of the rest of the rows. The node corresponding to the deleted row is called a *datum (reference) node*. The matrix obtained after deleting a dependent row is called an *incidence matrix* of S and it is denoted by **B**.

Although **A** and **B** are of great theoretical value, however, the storage requirements for these matrices are high and proportional to N×N and M×(N–1),

respectively. In fact a large number of unnecessary zeros is stored in these matrices. In practice one can use different approaches to reduce the storage required, some of which are described in the following.

Member List: This type of representation is a common approach in structural mechanics. A member list consists of two rows (or columns) and M columns (or rows). Each column (or row) contains the labels of the two end nodes of each member, in which members are arranged sequentially. For example, the member list of S in Figure 1.14 is:

$$\mathbf{ML} = \begin{bmatrix} \mathbf{n}_1 & \mathbf{m}_2 & \mathbf{m}_3 & \mathbf{m}_4 & \mathbf{m}_5 & \mathbf{m}_6 & \mathbf{m}_7 \\ \mathbf{n}_1 & \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 & \mathbf{n}_4 & \mathbf{n}_5 & \mathbf{n}_6 & \mathbf{m}_7 \\ \mathbf{3} & \mathbf{5} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{4} \end{bmatrix}.$$
(1-14)

It should be noted that a member list can also represent orientations on members. The storage required for this representation is $2 \times M$. Some engineers prefer to add a third row containing the member's labels, for easy addressing. In this case the storage is increased to $3 \times M$.

A different way of preparing a member list is to use a vector containing the end nodes of members sequentially; e.g. for the previous example this vector becomes:

$$(1,3;3,5;1,2;2,3;3,4;4,5;2,4).$$
 (1-15)

This is a compact description of a graph; however, it is impractical because of the extra search required for its use in various algorithms.

Adjacency List: This list consists of N rows and D columns, where D is the maximum degree of the nodes of S. The *i*th row contains the labels of the nodes adjacent to node i of S. For the graph S shown in Figure 1.14, the adjacency list is:

$$\mathbf{AL} = \begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \mathbf{n}_2 \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \\$$

The storage needed for an adjacency list is N×D.

Compact Adjacency List: In this list the rows of **AL** are continually arranged in a row vector **R**, and an additional vector of pointers **P** is considered. For example, the compact adjacency list of Figure 1.14 can be written as:

$$\mathbf{R} = (\mathbf{2}, 3, \mathbf{1}, 3, 4, \mathbf{6}, 2, 4, 5, \mathbf{2}, 3, 5, \mathbf{3}, 4),$$
$$\mathbf{P} = (1, 3, 6, 10, 13, 15). \tag{1-17}$$

P is a vector $(p_1, p_2, p_3, ...)$ which helps to list the nodes adjacent to each node. For node n_i one should start reading **R** at entry p_i and finish at $p_{i+1} - 1$.

An additional restriction can be put on \mathbf{R} , by ordering the nodes adjacent to each node n_i in ascending order of their degrees. This ordering can be of some advantage, an example of which is nodal ordering for bandwidth optimisation. The storage required for this list is 2M + N + 1.

1.5.2 CYCLE BASES MATRICES

The cycle-member incidence matrix $\overline{\mathbf{C}}$ of a graph S, has a row for each cycle or hinged cycle and a column for each member. An entry \overline{c}_{ij} of $\overline{\mathbf{C}}$ is 1 if cycle C_i contains member m_j and it is 0 otherwise. In contrast to the node adjacency and node-member incidence matrix, the cycle-member incidence matrix does not determine a graph up to isomorphism; i.e. two totally different graphs may have the same cycle-member incidence matrix.

For a graph S there exists $2^{b_1(S)} - 1$ cycles or hinged cycles. Thus \overline{C} is a $(2^{b_1(S)} - 1) \times M$ matrix. However, one does not need all the cycles of S, and the elements of a cycle basis are sufficient. For a cycle basis, a cycle-member incidence matrix becomes a $b_1(S) \times M$ matrix, denoted by C, known as the *cycle basis incidence matrix* of S. As an example, matrix C for the graph shown in Figure 1.14, for the following cycle basis,

$$C_1 = (m_1, m_3, m_4)$$
$$C_2 = (m_2, m_5, m_6)$$
$$C_3 = (m_4, m_5, m_7)$$

is given by:

The cycle adjacency matrix $\mathbf{D} = \mathbf{C}\mathbf{C}^{t} - \mathbf{W}$ is a $b_{1}(S) \times b_{1}(S)$ matrix, each entry d_{ij} of which is 1 if C_{i} and C_{i} have at least one member in common, and it is 0 otherwise.

For the above example,

$$\mathbf{D} = \mathbf{C}\mathbf{C}^{\mathsf{t}} - \mathbf{W} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$
(1-19)

where **W** is a diagonal matrix, in which a typical non-zero entry w_{ii} is the length of the cycle C_i . The trace of **CC**^t is equal to the total length of the cycles in the basis.

An important theorem can now be proved which is based on the orthogonality property mentioned in Section 1.4.8.

Theorem: Let S have incidence matrix \mathbf{B} and a cycle basis incidence matrix \mathbf{C} . Then:

$$\mathbf{CB}^{t} = \mathbf{0} \pmod{2}. \tag{1-20}$$

Proof: Consider the *i*th row of **C** and the *j*th column of **B**^t, which is the *j*th row of **B**. The *r*th entry in these two rows are both non-zero if and only if m_r is in cycle C_i and is incident with n_j . If m_r is in C_i , then n_j is also in C_i , but if n_j is in the cycle, then there are two members of C_i incident with n_j so that the (i,j)th entry of **CB**^t is $1+1 = 0 \pmod{2}$, and this completes the proof.

Matrix **C** for a fundamental cycle basis with special labels for its tree members and chords, finds a particular pattern. Let S have a tree T whose members are $M(T) = (m_1, m_2, ..., m_p)$ and a cotree for which $M(T^*) = (m_{p+1}, m_{p+2}, ..., m_{p+M(S)})$. Then there is a unique fundamental cycle C_i in $S - M(T^*) + m_i$, $p+1 \le i \le M(S)$ and this set of cycles forms a basis for the cycle space of S. As an example, for the graph S of Figure 1.12 (page 14) whose members are labelled as shown in Figure 1.15, the fundamental cycle basis consists of:

 $C_1 = (m_1, m_2, m_7)$ $C_2 = (m_2, m_3, m_8)$

 $C_3 = (m_1, m_4, m_9, m_5, m_2), \quad C_4 = (m_2, m_5, m_{10}, m_6, m_3).$

The corresponding C for the selected tree T is denoted by C_0 and has the following form:

$$C_{0} = \frac{C_{1}}{C_{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{T} & I \end{bmatrix}$$

$$M(T) \qquad M(T^{*}) \qquad (1-21)$$

Fig. 1.15 A graph and its tree members.

1.5.3 CUTSET BASES MATRICES

The *cutset-member incidence matrix* $\overline{\mathbf{C}}^*$ for a graph S, has a row for each cutset of S and a column for each member. An entry \overline{c}_{ij}^* of $\overline{\mathbf{C}}^*$ is 1 if cutset \mathbf{C}_i^* contains member m_j and it is 0 otherwise. This matrix like $\overline{\mathbf{C}}$ does not determine a graph completely. Independent rows of $\overline{\mathbf{C}}^*$ for a cutset basis, denoted by \mathbf{C}^* , form a matrix known as a *cutset basis incidence matrix*, which is a $\rho(S) \times M$ matrix, $\rho(S)$ being the rank of graph S. As an example, \mathbf{C}^* for the cutset of Figure 1.13 with members labelled as in Figure 1.15, is given below:

$$\mathbf{C}^{*} = \begin{bmatrix} \mathbf{C}_{1}^{*} & \mathbf{C}_{1} & \mathbf{C}_{2} & \mathbf{C}_{3} & \mathbf{C}_{4} & \mathbf{C}_{5} & \mathbf{C}_{6} & \mathbf{C}_{7} & \mathbf{C}_{8} & \mathbf{C}_{9} & \mathbf{C}_{10} \\ \mathbf{C}^{*} = \begin{bmatrix} \mathbf{C}_{2}^{*} & \mathbf{C}_{1} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{I} & \mathbf{I} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{I} \\ \mathbf{C}_{5}^{*} & \mathbf{C}_{6}^{*} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{bmatrix} \right].$$
(1-22)

The *cutset adjacency matrix* $\mathbf{D}^* = \mathbf{C}^* \mathbf{C}^{*t}$ is a $\rho(S) \times \rho(S)$ matrix defined analogously to cycle adjacency matrix \mathbf{D} .

For a fundamental cutset basis with appropriate labelling of the members in T and T^* , the particular pattern of C^* becomes:

$$\mathbf{C}_{0}^{*} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{C}_{c}^{*} \end{bmatrix}$$
(1-23)

From the orthogonality condition, $\mathbf{C}_0 \mathbf{C}_0^{*_t} = \mathbf{0}$, hence:

$$\begin{bmatrix} \mathbf{C}_{\mathrm{T}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_{\mathrm{c}}^{*_{\mathrm{t}}} \end{bmatrix} = \mathbf{0}.$$
 (1-24)

Hence $\mathbf{C}_{\mathrm{T}} + \mathbf{C}_{\mathrm{c}}^{*\mathrm{t}} = \mathbf{0} \pmod{2}$, resulting in :

$$\mathbf{C}_{\mathrm{T}} = \mathbf{C}_{\mathrm{c}}^{*\mathrm{t}} \,. \tag{1-25}$$

Therefore, for a graph having C_0 , one can construct C_0^* and vice versa.

There exists a very simple basis for the cutset space of a graph, which consists of N-1 cocycles of S. As an example, for the graph of Figure 1.14, considering n_5 as a datum node, we have,

$$\mathbf{C}^{*} = \begin{bmatrix} \mathbf{m}_{1} & \mathbf{m}_{2} & \mathbf{m}_{3} & \mathbf{m}_{4} & \mathbf{m}_{5} & \mathbf{m}_{6} & \mathbf{m}_{7} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$
(1-26)

which is the same as the incidence matrix \mathbf{B} of \mathbf{S} . The simplicity of the displacement method of structural analysis is due to the existence of such a simple basis.

1.6 DIRECTED GRAPHS AND THEIR MATRICES

An *oriented* or *directed* graph is a graph in which each member is assigned an orientation. A member is oriented from its initial node to its final node, as shown in Figure 1.16(a). The initial node is said to be positively incident on the member and the final node negatively incident, as shown in the Figure:



Fig. 1.16 An oriented member, a directed graph and a directed tree.

The choice of orientation of members of a graph is arbitrary; however, once it is chosen, it must be retained. Cycles and cutsets can also be oriented as shown in Figure 1.16(b).

As an example, m_7 is positively oriented in cycle C_i , and m_{10} is negatively oriented in cutset C_i^* .

All the matrices $\overline{\mathbf{B}}$, \mathbf{B} , \mathbf{C} and \mathbf{C}^* can be defined as before, with the difference of having +1, -1 and 0 as entries, according to whether the member is positively, negatively and zero incident with a cutset or a cycle.

As an example, for graph S in Figure 1.16(b) the matrix \mathbf{B} with n_1 as datum node is formed:

		m_1	m_2	m	^m 4	^m 5	^m 6	^m 7	^m 8	^m 9	^m 10	
	n ₂	-1	0	0	1	0	0	1	0	0	0]	
	n ₃	0	1	0	0	-1	0	-1	1	0	0	
п									-1			(1-27)
B =	n ₅	0	0	0	-1	0	0	0	0	1	0	(1 27)
	1								0			
	n ₇	0	0	0	0	0	1	0	0	0	-1	

Consider a tree as shown in Figure 1.16(c). The corresponding cycle basis incidence matrix can be written as:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{1} & 1 & 0 & 0 & 0 & 0 \\ \mathbf{C}_{2} & \mathbf{C}_{3} & \mathbf{C}_{4} & \mathbf{C}_{1} & 1 & 0 & 0 & 0 \\ \mathbf{C}_{3} & \mathbf{C}_{4} & \mathbf{C}_{1} & 1 & 0 & 1 & 1 & 0 \\ \mathbf{C}_{4} & \mathbf{C}_{1} &$$

Obviously

with a similar proof as that of the non-oriented case.

A cutset-member incidence matrix is similarly obtained as:

$$\mathbf{C}^{*} = \begin{bmatrix} \mathbf{C}^{*}_{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{2} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{3} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{4} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{5} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{*}_{5} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{C}^{*}_{6} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{C}^{*}_{7} & \mathbf{C}^{*}_{6} \end{bmatrix} \left[\begin{array}{c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0}$$

It can easily be proved that:

$$\mathbf{C}_{\mathrm{T}} = -\mathbf{C}_{\mathrm{c}}^{*\mathrm{t}} \,. \tag{1-31}$$

1.7 GRAPHS ASSOCIATED WITH MATRICES

Matrices associated with graphs are discussed in the previous sections. Sometimes it is useful to consider the reverse of this process and think of the graph associated with an arbitrary matrix **H**. Such a graph has a node associated with each row of the matrix and if h_{ij} is non-zero, then there is a connecting member from node i to node j. In the case of a symmetric matrix, there is always a connection from i to j whenever there is one from j to I; therefore one can simply use undirected

members. Two simple examples are illustrated in Figure 1.17 and Figure 1.18. The directed graph associated with a non-symmetric matrix is usually called a *digraph* and the word graph is used for the undirected graph associated with a symmetric matrix:



Fig. 1.17 A non-symmetric matrix H and its associated digraph.



Fig. 1.18 A symmetric matrix H and its associated graph.

For \mathbf{H} to be viewed as the adjacency matrix, due to the presence of diagonal entries, one loop should be added to each node. However, since the structural models have always non-zero diagonal entries and contain no loops, this addition is disregarded.

With a rectangular matrix **E** a bipartite graph S = (A,B) can be associated. For each row of **E** a node of A and with each column of **E** a node of B is associated. Two nodes of A and B are connected with a member of S if e_{ij} is non-zero. An example of this is shown in Figure 1.19.



Fig. 1.19 A rectangular matrix E and its associated bipartite graph.

A weighted bipartite graph can be defined for any $m \times n$ rectangular matrix **H**. The nodes $r_1, r_2, ..., r_m$ and nodes $c_1, c_2, ..., c_n$ of the graph correspond to rows and columns of **H**, respectively. If $h_{ij} \neq 0$, then r_i is joined to c_j by a member whose weight is h_{ij} .

1.8 PLANAR GRAPHS - EULER'S POLYHEDRA FORMULA

Graph theory and properties of planar graphs, were first discovered by Euler in 1736. After 190 years Kuratowski found a criterion for a graph to be planar. Whitney developed some important properties of embedding graphs in the plane. MacLane expressed the planarity in terms of the graph's cycle basis. In this section some of these criteria are studied, and Euler's polyhedra formula is proven.

1.8.1 PLANAR GRAPHS

A graph S is called *planar* if it can be drawn (embedded) in the plane in such a way that no two members cross each other. As an example, a complete graph K_4 shown in Figure 1.20 is planar since it can be drawn in the plane as shown:



Fig. 1.20 K_4 and two of its drawings.

On the other hand K_5 , Figure 1.21, is not planar, since every drawing of K_5 contains at least one crossing.



(a) K_5 .

(b) Two drawings of K_5 with one crossing.



Similarly K_{3,3}, Figure 1.22, is not planar, as illustrated:



(a) K_{3,3}.
 (b) Two drawings of K_{3,3} with one crossing.
 Fig. 1.22 K_{3,3} and its drawings.

A planar graph S drawn in the plane divides the plane into regions all of which are bounded and only one is unbounded. If S is drawn on a sphere, all the regions will be bounded; however, the number of regions will not change. The cycle bounding a region is called a *regional cycle*. Obviously the sum of the lengths of regional cycles is twice the number of members of the graph.

There is an outstanding formula that relates the number of regions, members and nodes of a planar graph, in the form,

$$R(S) - M(S) + N(S) = 2$$
,

where R(S), M(S) and N(S) are the numbers of regions, members and nodes of planar graph S, respectively. This formula shows that for different drawings of S in the plane, R(S) remains constant.

Originally the above relationship was given for polyhedra, in which R(S), M(S) and N(S) correspond to faces, edges and corners of a polyhedron, respectively. However, the theorem can easily be expressed in graph-theoretical terms as follows.

Theorem (Euler [45]): Let S be a connected planar graph. Then:

$$R(S) - M(S) + N(S) = 2.$$
 (1-32)

Proof: For proof, S is reformed in two stages. In the first stage, a spanning tree T of S is considered in the plane for which R(T) - M(T) + N(T) = 2. This is true since R(T) = 1 and M(T) = N(T) - 1. In second stage chords are added one at a time. Addition of a chord increases the number of members and regions each by

unity, leaving the left hand side of Eq. (1-32) unchanged during the entire process, and the result follows.

1.8.2 THEOREMS FOR PLANARITY

In order to check the planarity of a graph, different approaches are available which are based on the following theorems. These theorems are only stated and the reader may refer to textbooks on graph theory for proofs.

Theorem (Kuratowski [148]): A graph S is planar if and only if it has no subgraph contractible to K_5 or $K_{3,3}$.

Contracting a member $m_k = (n_i, n_j)$ is an operation in which the member is removed and n_i is identified with n_j so that the resulting node is incident to all members (other than m_k) that were originally incident with n_i or n_j . If a graph S' can be obtained from S by succession of member contractions, then S is *contractible* to S'. The process of the contraction of a member (n_i, n_j) of a graph is shown in Figure 1.23(a), and the contraction of the Petersen graph to K_5 is illustrated in Figure 1.23(b).



(a) Contraction of a member m_k .



(b) Contraction of the Petersen graph to K₅.

Fig. 1.23 The contraction of a member in a graph.

Theorem (MacLane [167]): A connected graph is planar if and only if every block of S with at least three nodes has a cycle basis, $C_1, C_2, ..., C_{b_1(S)}$ and one additional cycle C_0 , such that every member is contained in exactly two of these $b_1(S)+1$ cycles.

A *block* is a maximal non-separable graph, and a *non-separable* graph is a graph that has no cut-points. A *cut-point* is a node whose removal increases the number of components and a bridge is such a member. In Figure 1.24, a graph and its blocks are illustrated:



Fig. 1.24 A graph and its blocks.

Definitions: A graph S^* is a dual graph of a graph S if there is a 1-1 correspondence between the members of S^* and those of S, such that a set of members in S^* is a cycle vector of S^* if and only if the corresponding set of members in S is a cutset vector of S.

Theorem (Whitney [250]) - A graph is planar if and only if it has a combinatorial dual.





For a connected planar graph S, the dual graph S* is constructed as follows: To each region r_i of S there is a corresponding node r_i^* of S* and to each member m_j of S there is a corresponding member m_j^* in S*, such that if the member m_j occurs on the boundary of two regions r_1 and r_2 , then the member m_j^* joins the corresponding nodes r_1^* and r_2^* in S*, Figure 1.25.

1.9 MAXIMAL MATCHING IN BIPARTITE GRAPHS

1.9.1 DEFINITIONS

As defined before, a graph is bipartite if its set of nodes can be partitioned into two sets A and B, such that every member of the graph has one end node in A and other in B. Such a graph is denoted by S = (A,B). A set of members of S is called a *matching* if no two members have a common node. The size of any largest matching in S is called the *matching number* of S, denoted by $\psi(S)$. A subset N'(S) \subseteq N(S), is the *node cover of* S, if each member of S has at least one end node in N'(S). The cardinality of any smallest node cover, denoted by $\tau(S)$, is known as the *node covering number* of S.

1.9.2 THEOREMS ON MATCHING

In this section, three theorems are stated, and the proofs may be found in the book by Lovasz and Plumner [164]:

Theorem 1 (König [141,142]): For a bipartite graph S, the matching number $\psi(S)$ is equal to the node covering number $\tau(S)$.

Theorem 2 (Hall [69]): Let S = (A,B) be a bipartite graph. Then S has a complete matching of A into B if and only if $|\Gamma(X)| \ge |X|$ for all $X \subseteq A$.

 $\Gamma(X)$ is the image of X, i.e. those elements of B which are connected to the elements of X in S. Figure 1.26(a) shows a bipartite graph for which matching exists and Figure 1.26(b) illustrates a case where complete matching does not exist, because $X = (a_1, a_2)$ are matched to b_1 , i.e. $|\Gamma(X)| \le |X|$:



Fig. 1.26 Matching in bipartite graphs.

A perfect matching is a matching which covers all nodes of S.

Theorem 3 (Frobenius [53]): A bipartite graph S = (A,B) has a perfect matching if and only if |A| = |B| and for each $X \subseteq A$, $|\Gamma(X)| \ge |X|$.



(a) Perfect matching exists. (b) Perfect matching does not exist.

Fig. 1.27 Perfect matching in bipartite graphs.

This is also known as the marriage theorem. Figures 1.27(a) and (b) show cases when a perfect matching exists and does not exist, respectively.

Therefore Frobenius's theorem characterizes those bipartite graphs which have a perfect matching. Hall's theorem characterizes those bipartite graphs that have a matching of A into B. König's theorem gives a formula for the matching number of a bipartite graph.

1.9.3 MAXIMUM MATCHING

Let M be any matching in a bipartite graph S = (A,B). A path P is called an *alternating path with respect to M*, or an *M-alternating path* if its members (edges) are alternately chosen from the matching M and outside M. A node is *exposed* (unmatched, not covered) with respect to matching M if no member of M is incident with that node. An *alternating tree* relative to the matching, is a tree which satisfies the following two conditions: first, the tree contains exactly one exposed node from A, which is called its *root*, second, all paths between the root and any other node in the tree are alternating paths.

As an example, in Figure 1.28(a) the path $a_1b_1a_3b_3a_4$ is an alternating path with respect to the matching shown in bold lines; a_2 , b_2 , a_4 and b_4 are exposed nodes.



(a) An arbitrary matching. (b) An augmented matching.

Fig. 1.28 Operation for maximum matching.

An M-alternating path joining two exposed nodes is called an *M*-augmenting path. For every such path the corresponding matching can be made larger by discarding the members of $P \cap M$ and adding those of P - M, where P is an M-alternating path $b_2a_1b_1a_3b_3a_4$, Figure 1.28(b). Thus, if S contains any M-alternating path P joining two exposed nodes, then M can not be a maximum matching, for one can readily obtain a larger matching M' by discarding the members of $P \cap M$ and adding those of P - M.

Theorem 4 (Berge [13-14]): Let M be a matching in a graph S. Then M is a maximum matching if and only if there exists no augmenting path in S relative to M.

The above result provides a method for finding a maximum matching in S. The computational procedure for construction of a maximum matching begins with considering any feasible matching, possibly the empty matching. Each exposed node of A is made the root of an alternating tree, and nodes and members are added to the trees by means of a labelling technique. Eventually, the following two cases must occur: either an exposed node in B is added to one of the trees, or else

it is not possible to add more nodes and members to any of the trees. In the former case, the matching is augmented and the formation of trees is repeated with respect to the new matching. In the latter case, the trees are said to be *Hungarian* and the process is terminated.

As an example, consider the matching shown in Figure 1.29(a), in which bold lines represent members in the matching. Alternating trees are constructed, with the exposed nodes a_1 and a_5 of A as roots, as shown in Figure 1.29(b). An augmenting path is found, as indicated in the Figure. Naturally, several different sets of alternating trees could have been constructed. For example, the tree rooted at node a_1 could have contained the member (a_2,b_3) .



Fig. 1.29 A bipartite graph and its alternating tree.

Augmented matching is shown in Figure 1.30. When the alternating tree of Figure 1.31 is used for the augmented matching, it becomes Hungarian.



Fig. 1.30 Augmented matching.



Fig. 1.31 Alternating tree for augmented matching.

BIPARTITE MATCHING ALGORITHM

Let X be any matching, possibly the empty matching of a bipartite graph S = (A,B). No nodes are labelled.

Step 1 (labelling):

- 1.1 Give the label \emptyset to each exposed node in A.
- 1.2 If there are no unscanned labels, go to Step 3. Otherwise, find a node i with an unscanned label. If $i \in A$, go to Step 1.3; if $i \in B$, go to Step 1.4.
- 1.3 Scan the label on node i $(i \in A)$ as follows: for each member $(i,j) \notin X$ incident to node i, give node j the label "i", unless node j is already labelled. Return to Step 1.2.
- 1.4 Scan the label on node i (i∈B) as follows: if node i is exposed, go to Step 2. Otherwise, identify the unique member (i,j)∈X incident to node i and give node j the label "i". Return to Step 1.2.

Step 2 (Augmenting):

An augmenting path has been found, terminating at node i (identified in Step 1.4). The nodes preceding node i in the path, are identified by backtracking. That is, if the label on node i is "j", the second-to-last node in the path is j. If the label on node j is "k", the third-to-last node is k, and so on. The initial node in the path has the label " \emptyset ". Augment X by adding to X all members in the augmenting path that are not in X and removing those which are in X. Remove all labels from nodes. Return to Step 1.1.

Step 3 (Hungarian Labelling):

The labelling is Hungarian, no augmenting path exists, and the matching X is of maximum cardinality.

For further study, the reader may refer to the original paper of Hopcroft and Karp [80] or Lawler [157]. An algorithm using a different approach may be found in Ref. [6].

EXERCISES

1.1 In the following graph, which members are incident with node 3 (identify with their end nodes)? Which nodes are adjacent to node 4? What is the degree of node 2?



1.2 Are the following graphs isomorphic?



1.3 Draw a tree, a spanning tree and an SRT rooted at O for the following graph. Use O' as the root of a second SRT and compare its length and width with those of the first one.



1.4 What types of graph are the following?



1.5 List all the cycles of the following graph:



1.6 Prove that for a planar graph embedded on a sphere with all triangular faces, M(S) = 3N(S) - 6.

1.7 Find a fundamental cycle basis of the following graph using an arbitrary spanning tree:

	0	
,		

1.8 In the above example use two SRTs rooted at O and O' and compare the length of the corresponding fundamental cycle bases.

1.9 Write the adjacency and member-node incidence matrices of the graphs in Exercise 1.2. Use an arbitrary node and member numbering. What can you say about the resulting matrices?

1.10 Write C, C*, C₀ and C^{*}₀ matrices for the following graph and examine the orthogonality property:



1.11 Identify the planar graphs in the following figure:



1.12 Prove that K_5 and $K_{3,3}$ are not planar.

1.13 Euler's formula as in Eq. (1-32) fails for disconnected graphs. If a planar graph S has $b_0(S)$ components, how can the formula be adjusted?

1.14 Find a maximal matching for the following bipartite graphs. Which one is complete and which one is a perfect matching?



1.15 Why is there no complete matching for the following bipartite graphs?

CHAPTER 1 Basic Concepts and Definitions

