Chapter 2:

Simple Beams and Frame; Derivation of Plastic Load Factor

Introduction

Collapse of a simply supported beam

Collapse of a redundant beam

Construction of bending moments diagram for plastic analysis

Traveling loads

The work equation

Rectangular portal frames

Trial and error graphic method for design

Pitched roof portal frames

2
SIMPLE BEAMS AND FRAMES; ANALYSIS AND DESIGN

2.1 INTRODUCTION

The ease and power of plastic methods become apparent when they are applied to redundant structures. In this chapter further idealisation is considered. The shape factor is considered to be unit, then M–k curve becomes as shown in Fig. 2. and

$$M_p = M_Y$$

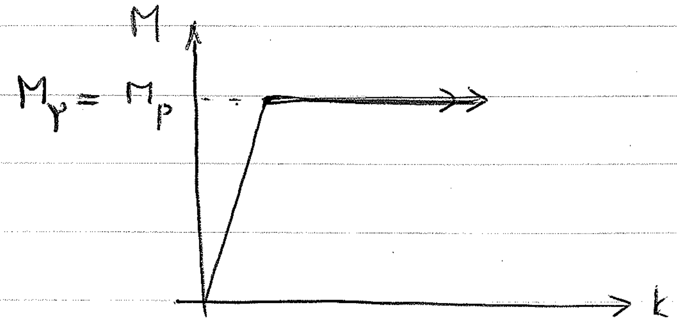


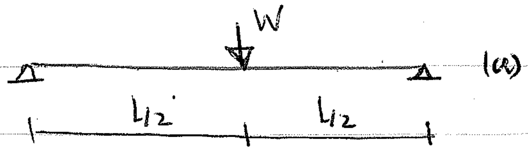
Fig. 2.1 Idealised M-k curve. Corresponding to $\nu = 1$

This idealisation, while simplifying the numerical work, does not restrict the validity of the arguments; it will be seen that the final collapse load of a frame depends only on the value of M_p and not on the complete M–K relationship.

2.2 COLLAPSE OF A SIMPLY SUPPORTED BEAM

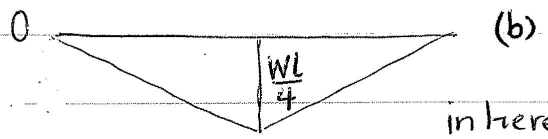
We study a simply supported beam from start of loading until it collapses.

a) Loading



b) Bending moment diagram

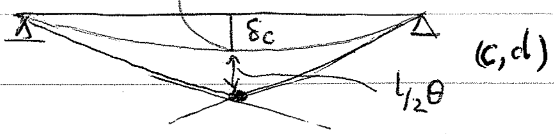
(Sagging bending moment considered negative)



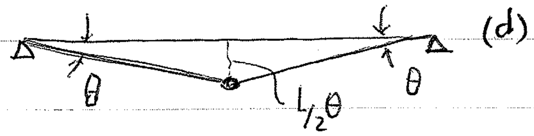
in here $\frac{WL}{4} = M_p$ and a hinge is formed.

c) i) Elastic deflected form $W = W_c$

ii) Deflected form during Collapse



d) Changes of deflection during collapse



Plastic collapse load:

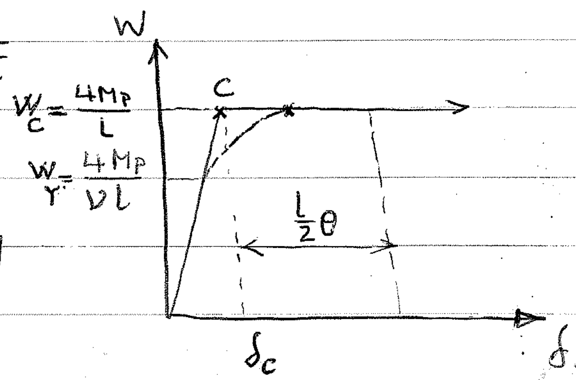
$$\frac{1}{4} W_c L = M_p \quad \rightarrow \quad W_c = \frac{4M_p}{L}$$] statical approach

Since the bending moment at every cross section is less than M_p, the beam remains elastic everywhere except at the Centre.

The elastic central deflection δ of the beam is $\frac{WL^3}{48EI}$.
As the collapse load is attained the central deflection δ_c at the point of collapse is therefore given by

$$\delta_c = \frac{W_c L^3}{48EI} = \frac{M_p L^2}{12EI}$$

The behaviour of beam can be summerised on a diagram relating W to the central deflection.



The broken curve is the effect of taking into account the difference between M_y and M_p. Elastic behaviour would cease at the yield load W_y when the central B.M. was M_y, where

$$W_y = \frac{4M_y}{L} = \frac{4M_p}{\nu L} = \frac{W_c}{\nu} \quad]$$

ν being the shape factor. PLASTIC COLLAPSE would still occur at the same value of W as before, but greater deflections would be developed before collapse.

The ratio of $\frac{W_c}{W_y} = \nu$ for a statically determinate structure, in which the greatest bending moment is proportional to the load and occurs at the same position regardless of the value of load. Yield occurs when this greatest bending moment is equal to M_y, and collapse occurs when it is equal to M_p, for the introduction of a single hinge is always sufficient to reduce a stat. deter. str. to a mechanism. It follows that the ratio of W_c to W_y is the same as the ratio of M_p to M_y

W_c can also be found by kinematic approach, Horne (1949).

During collapse there is no change in the elastic strain energy stored in the beam, since the B.M. distribution stays unaltered. The work done by the loads during a small motion of collapse mechanism is therefore equal to the work absorbed in the plastic hinge. The load W_c moves through a distance $\frac{L\theta}{2}$ and so does work $W_c\frac{L\theta}{2}$. The rotation at hinge is 2θ, so the work absorbed in the hinge is $2M_p\theta$. Thus

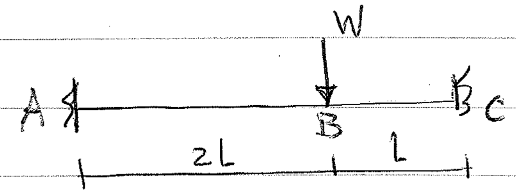
$$\frac{1}{2}W_c L\theta = 2M_p\theta \quad \rightarrow \quad W_c = \frac{4M_p}{L}$$

A Kinematical approach.

2.3 collapse of a redundant beam

Consider a beam with Fixed ends as shown in Fig. 2. subjected to a point load W. This load increases slowly until collapse occurs. For small M the behaviour is elastic and B.M. is as shown

First consider an elastic analysis of Beam to compare with plastic analysis

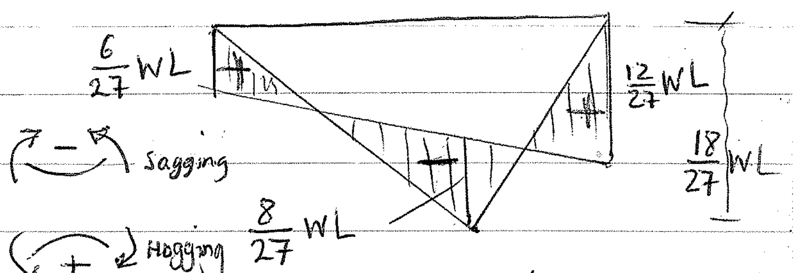


ELASTIC SOLUTION:

$\gamma(s) = 2$

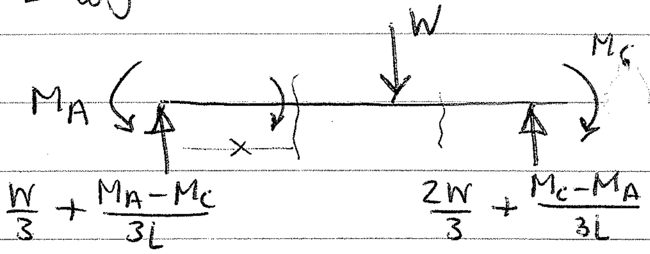
Redundants M_A, M_C

Two compatibility at two ends should be satisfied using



$$M = EI\,k = EI\,\frac{d^2y}{dx^2}$$

$$M_x = M_A - \left(\frac{W}{3} + \frac{M_A - M_C}{3L}\right)x + W\langle x-2L\rangle$$



$$EI\frac{d^2y}{dx^2} = M_A - \left(\frac{W}{3} + \frac{M_A - M_C}{3L}\right)x + W\langle x-2L\rangle$$

$\langle\,\rangle$ means ignore if negative

$$EI\frac{dy}{dx} = M_A x - \left(\frac{W}{3} + \frac{M_A - M_C}{3L}\right)\frac{x^2}{2} + \frac{1}{2}W\langle x-2L\rangle^2 + A$$

$$EIy = \frac{M_A}{2}x^2 - \left(\frac{W}{3} + \frac{M_A - M_C}{3L}\right)\frac{x^3}{6} + \frac{W}{6}\langle x-2L\rangle^3 + Ax + B$$

Four unknowns M_A, M_C, A and B

B. Cs. $\quad y(0) = 0 \quad y'(0) = 0 \quad y(3L) = 0 \quad y'(3L) = 0$

leading to $\quad M_A = \frac{6}{27}WL = \frac{2}{9}WL, \; M_C = \frac{12}{27}WL = \frac{4}{9}WL$

For elastic solution we need

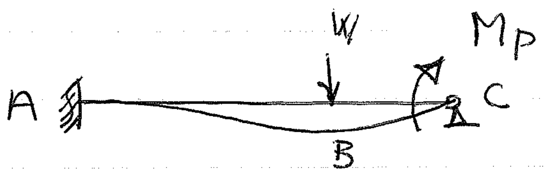
COMPATIBILITY (Displacement Condition)
EQUILIBRIUM (Statics)
MOMENT – CURVATURE RELATION ($M = -EIK$)

Elastic PLASTIC ANALYSIS

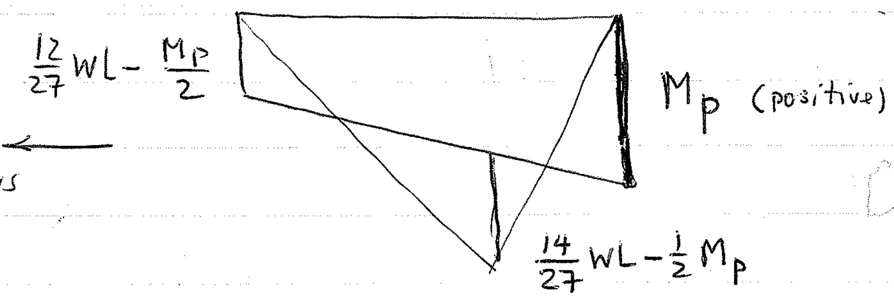
Elastic bending moment is valied if the largest B.M. $M_C = \frac{12}{27} WL < M_p$. As the load increases a plastic hinge is formed at C at the load

$\frac{12}{27} WL = M_p \rightarrow \boxed{W = 9M_p/4L}$

M_p is a restraining moment and opposite direction of rotation



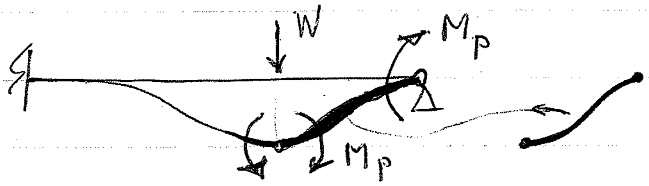
For these values similar solution to Elastic analysis is performed.



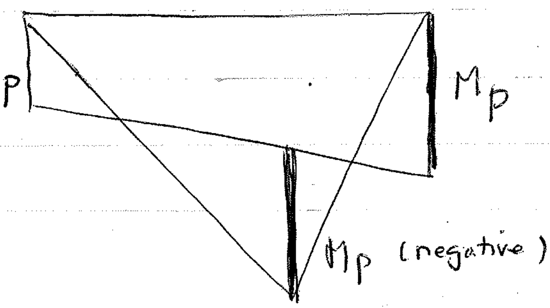
Further increase will produce a second hinge at B

This is when $\boxed{W = \frac{81 M_p}{28 L}}$

$\frac{14}{27} WL - \frac{1}{2} M_p = M_p$



For these values Equilibrium $2WL - 5M_p$ is enough since the structure is determinate

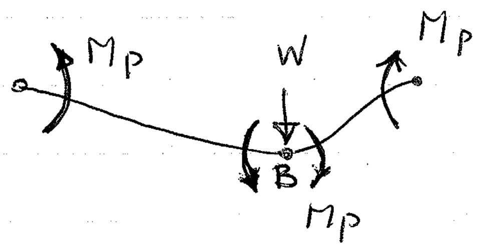


$2WL - 5M_p = M_p$

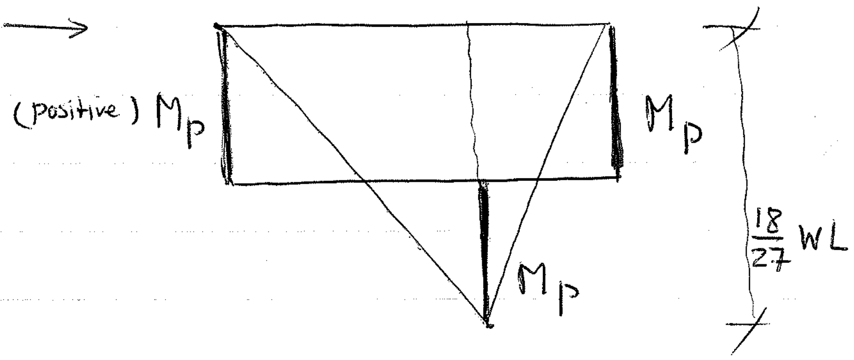
$\boxed{W = \frac{6M_p}{2L} = \frac{3M_p}{L}}$

When W is increased to $\frac{3M_P}{L}$, a final hinge is formed at A, making the beam a mechanism.

i.e. $\boxed{W_c = \frac{3M_P}{L}}$

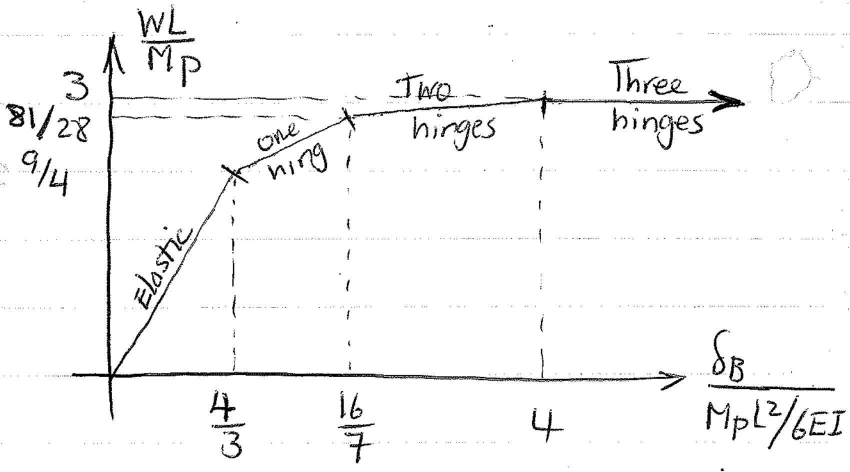


These values are obvious. →



For each of the above cases one can compute deflections from differential equation of Bending. Plotting W against the deflexion of the loading point B, the following curve is obtained.

Fig. 2.1 Loading history of the beam



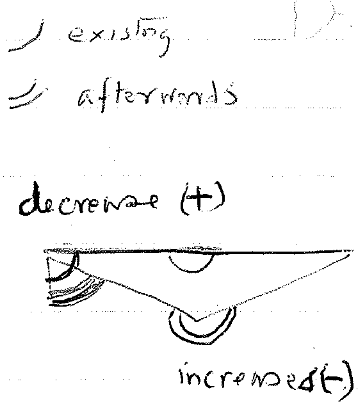
IMPORTANT NOTES:

1. Sharp corners is due to assumption made for material i.e. elastic-perfectly plastic moment-curvature relationship. If releastic curve is used the above corners will be rounded.

2. The final collapse load $W_c = \frac{3M_p}{L}$ depends on the value of M_p alone; while the precise form of the load-deflexion curve is affected by the form of M-K, it is only the value of M_p which is used in writing the collapse Equation.

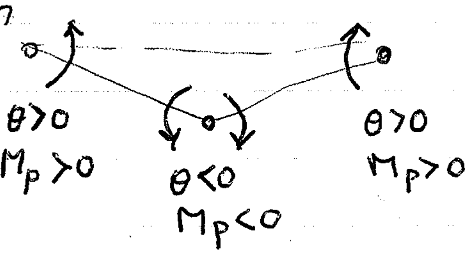
3. Final value of collapse load is independent of the order in which hinges are formed. Thus tracing of the complete loading history of the beam is not needed for the purpose of simple plastic analysis (or design).

4. Thus a structure needs to be examined at its collapse state and a step-by-step approach is not needed. Therefore, plastic analysis is more simple than elastic analysis.



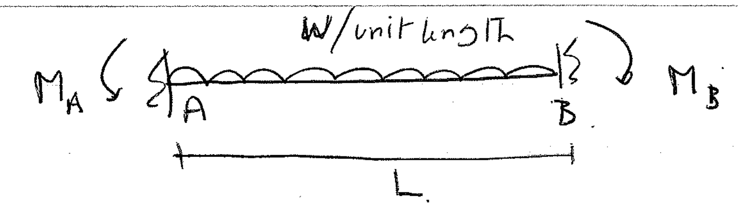
NOTE ABOUT SIGN CONVESION

M_p alway opposes rotation θ, e.g.
But M_p has the same sign as θ



Therefore, once sign for θ is defined every thing becomes clear.
For such as sign increase (for -) and decrease (+) of the angle is useful.

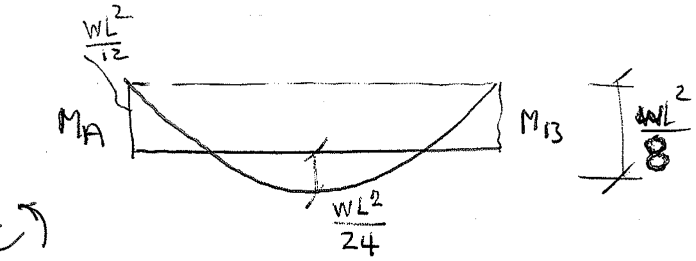
ANOTHER EXAMPLE

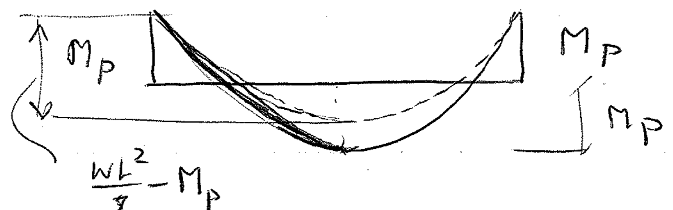


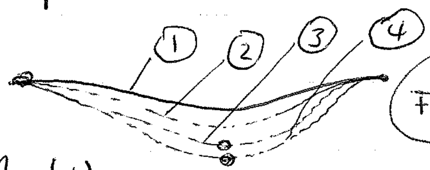
First we obtain an elastic solution:

EQUILIBRUIM

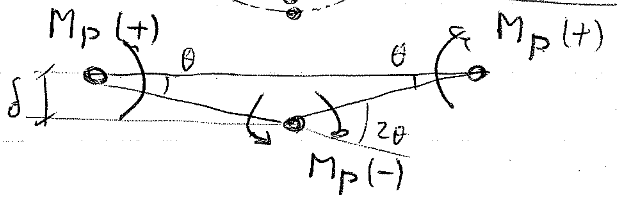
Equilibrium is satisfied by drawing the bending moment as a superposition of the simply supported sagging parabolic moment (with peak value $wL^2/8$) on the uniform moment diagram due to the end hogging moments M_A and M_B.







See The Fig. next page



MOMENT - CURVATURE RELATION

By any standard method this is deduced (Say from table or Slope-defl EQ)

$$\theta_A = \frac{wL^3}{24EI} - \frac{M_A L}{3EI} - \frac{M_B L}{6EI} \qquad \theta_A \text{ Clockwise}$$

$$\theta_B = \frac{wL^3}{24EI} - \frac{M_A L}{6EI} - \frac{M_B L}{3EI} \qquad \theta_B \text{ anti clockwise}$$

$$\Delta_C = \frac{5\, wL^4}{384\, EI} - \frac{M_A L^2}{16EI} - \frac{M_B L^2}{16EI}$$

COMPATIBILITY CONDITION

$$\theta_A = \theta_B = 0$$

Solution gives the well-known result $M_A = M_B = \frac{wL^2}{12}$

Equil. leads to

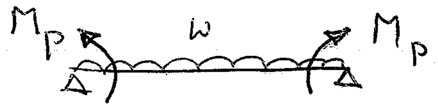
$$M_C = \frac{wL^2}{8} - \left[\frac{M_A}{2} + \frac{M_B}{2}\right] = \frac{wL^2}{24}$$

$$\Delta_C = \frac{wL^4}{384 EI}$$

Increase the load intensity until hinges are formed at A and B, i.e.

$$\frac{WL^2}{12} = M_p$$

i.e. $$\boxed{W = \frac{12 M_p}{L^2}}$$



A sligth load increase causes the plastic hinges to rotate while sustaining constant M_p. i.e beam behaves after this like a simply supported beam

$$\Delta_C = \frac{5}{384}\left(\frac{12 M_p}{L^2}\right)\frac{L^4}{EI} - \frac{M_p L^2}{16EI} - \frac{M_p L^2}{16EI} = \frac{M_p L^2}{32EI}$$

and central sagging M. = $M_c = \frac{WL^2}{8} - \left[\frac{M_p}{2} + \frac{M_p}{2}\right] = \frac{WL^2}{8} - M_p$

Further increase produces a third hinge at mid-span. i.e M_c becomes M_p

Hence

$$M_p = \frac{W_c L^2}{8} - M_p$$

$$M_p = \frac{W_c L^2}{16} \qquad \boxed{W_c = \frac{16 M_p}{L^2}}$$

central deflection can also be found

$$\Delta_C = \frac{M_p L^2}{12 EI}$$

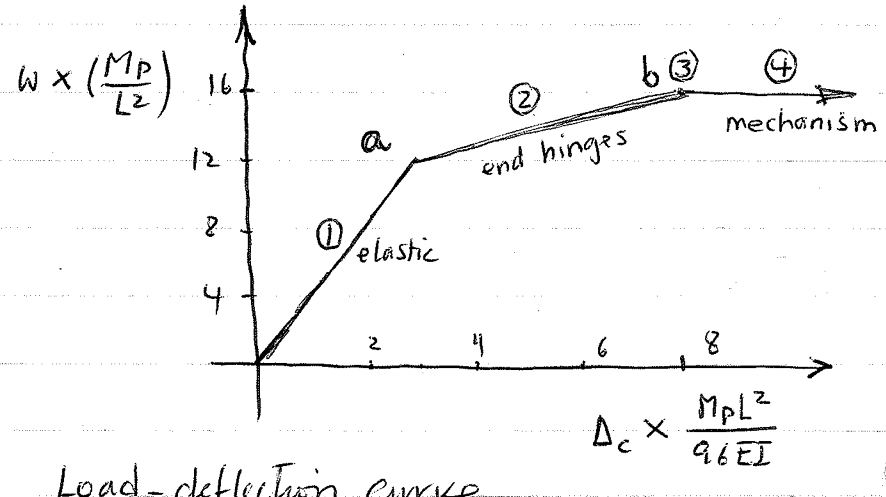


Fig. 2 Load-deflection curve for fixed ended beam with distributed load.

Load-displacement Diagram of a portal

This is a flexible portal with L = 508 mm. Cross section is b×d with

b = 12.7 mm d = 3.175 mm

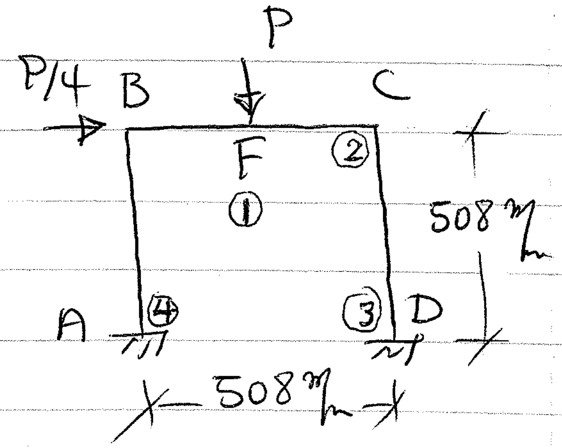
$E = 207\ kN/mm^2$

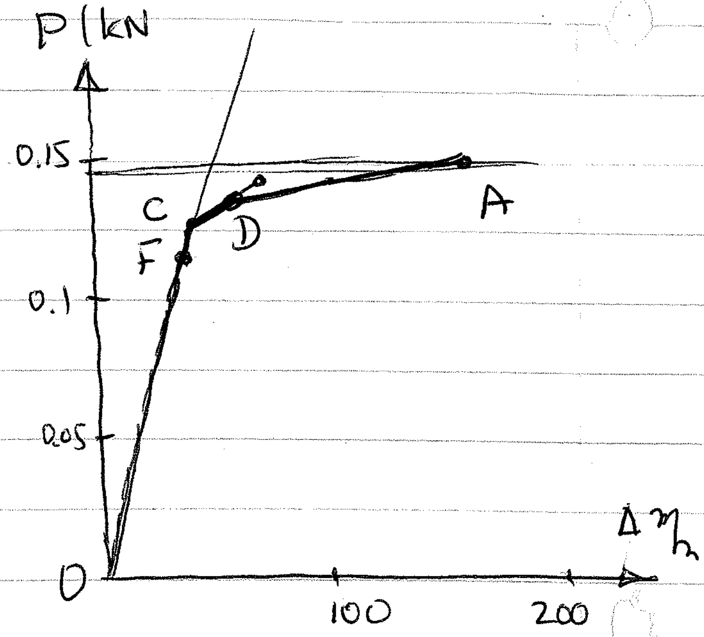
$M_p = 9.27\ kN.mm$

Shape factor = 1

Denoting $\frac{EI}{L} = k$

Δ = horizontal dispt.





$M_{AB} = 2k\theta_B - 6k\Delta/L$

$M_{BA} = 4k\theta_B - 6k\Delta/L$

$M_{BC} = 4k\theta_B + 2k\theta_C - \frac{PL}{8}$

$M_{CB} = 4k\theta_C + 2k\theta_B + \frac{PL}{8}$

$M_{CD} = 4k\theta_C - 6k\Delta/L$

$M_{DC} = 2k\theta_C - 6k\Delta/L$

Equilibrium EQS:

$\Sigma M_B = M_{BA} + M_{BC} = 0$

$8k\theta_B + 2k\theta_C - 6k\Delta/L = PL/8$

$\Sigma M_C = 0$

$2k\theta_B + 8k\theta_C - 6k\Delta/L = -\frac{PL}{8}$

Finally the shear EQ. leads to

$\frac{PL}{4} + M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$

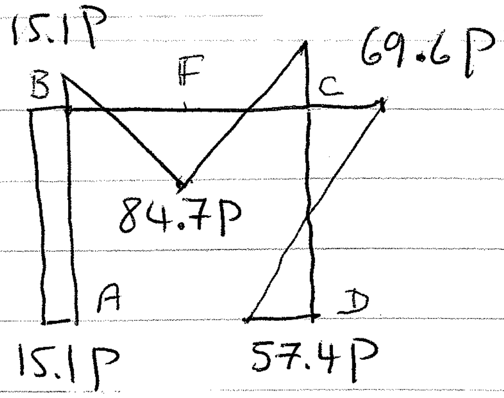
or $\frac{PL}{4} + 6k\theta_B + 6k\theta_C - 24k\Delta/L = 0$

$\Rightarrow$ $\Delta = \frac{5PL^2}{336k}$

$\theta_B = \frac{5PL}{168k}$

$\theta_C = -\frac{PL}{84k}$

Bending Moment Diagram

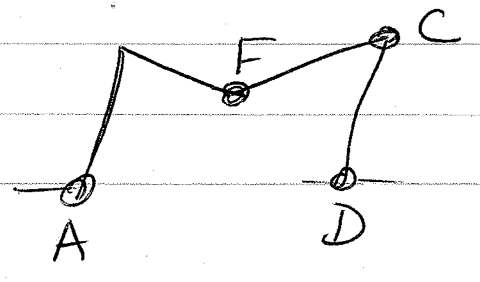


First hinge will be formed under the vertical load when load is increased Such That 84.7P changes to 9.27×10^{-3} kN.m This Corresponds to P=0.109 kN i.e point C in the P–Δ curve.

After C we have a hinge at point F and plastic moments M_p opposing the rotation. NOW again Slope-deflection can be applied

Unknowns: $\theta_B, \theta_C, \Delta, \theta_{FB}, \theta_{FC}, \delta$
where δ is The vertical dispt. of F
F can be treated as a joint for the frame and solution results in The value of P=0.120 kN with $\Delta = 33$ mm for the formation of the second hinge at C.

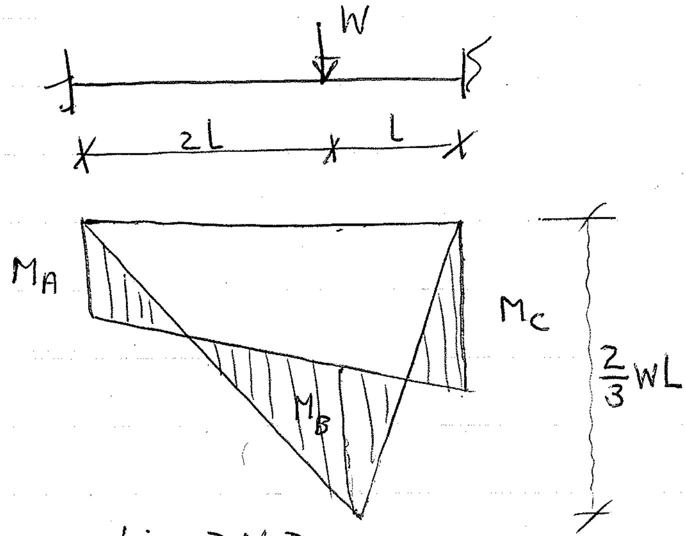
The above procedure is repeated to find the P for the formation of the third hinge D and Then The fourth hinge at A and Collapse



~~2.4 CONSTRUCTION OF BENDING MOMENT~~ DIAGRAM FOR PLASTIC ANALYSIS

Obivronsly under applied load W moments will be induced at two ends and we have the following B.M.D.

Graphical representation → of the applications of statics



By decomposition and drawing Corresponding B.M.Ds

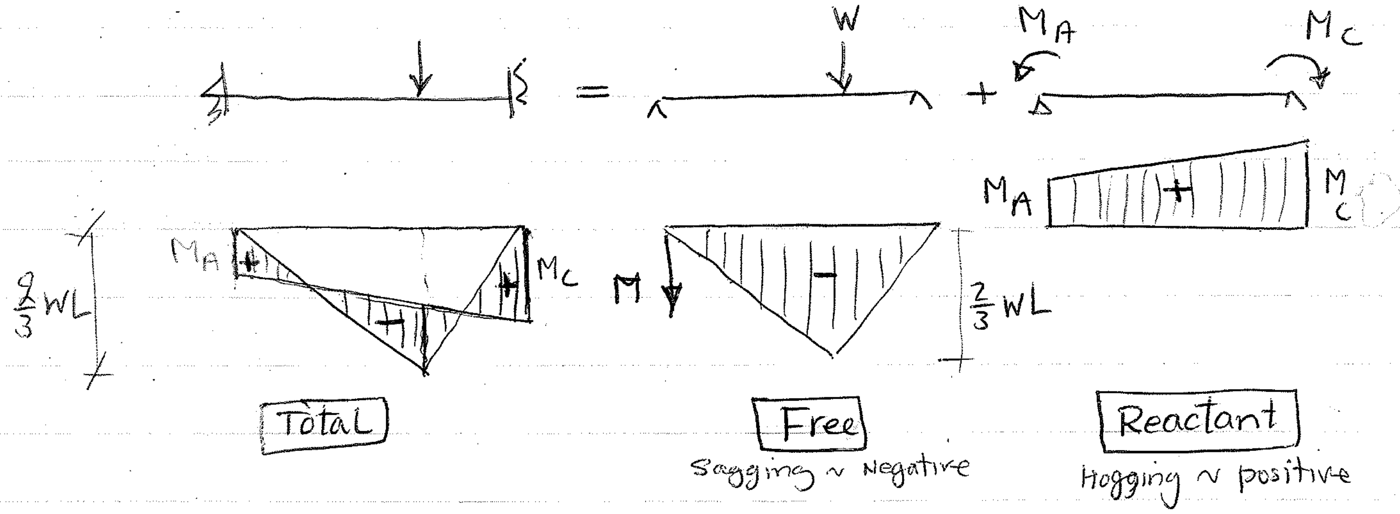


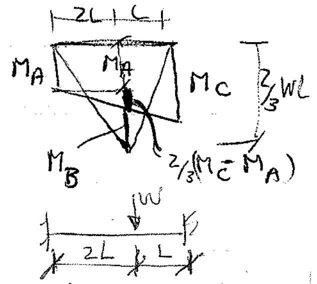
Fig. 2. Graphical construction of B.M.D.

By addition of free and reactant B.M.Ds The total B.M.D is obtained

Graphical addition is no more that analytical addition which occurs in the equations for the bending moments

2.4.1 POSITIONING THE REACTANT LINE

The essential problem in elastic or plastic problems consists of the positioning of the reactent line. In Fig. 2. some relation exists between M_A, M_B and M_C.



$$M_B + M_A + \frac{2}{3}(M_C - M_A) = \frac{2}{3}WL$$

or

$$M_A + 3M_B + 2M_C = 2WL$$

If any two of three B.Ms are known, the third one can be calculated, but there is no way by static alone of finding these two values. Fig. 2. is a graphical representation of the application of statics to the beam under the action of load W, and the beam is twice indeterminate.

In elastic analysis we needed compatibility condition, Similar extra conditions are required for plastic solution, but these are much simpler than the corresponding elastic statements, which can be deduced by inspection of Fig. 2. That is, we only need sufficient plastic hinges to form a mechanism of collapse. One can position reactant line so that $M_A = M_B = M_C = M_P$ and directly from diagram, or from above equation

$$6M_P = 2W_cL \quad \text{or} \quad 2M_P = \frac{2}{3}W_cL \quad \text{(The same)}$$

By inspection it is obvious that B.M. will no where be more than M_P, i.e. Yield condition is also satisfied.

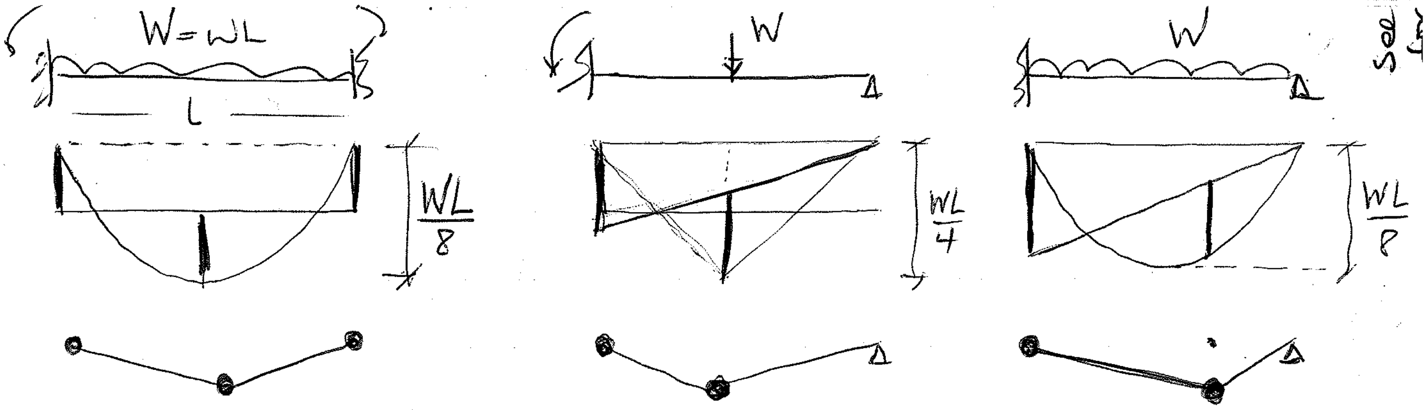
Thus 3 master requirements of plastic analysis and design are

$$\begin{cases} \text{MECHANISM} \\ \text{EQUILIBRIUM} \\ \text{YIELD } |M| \le M_P \end{cases}$$

2.4.2 ~~GRAPHICAL ANALYSIS AND DESIGN~~

Simple examples are analysed by plastic method. unlike elastic approach, The analyses are very simple

See page 21 for ...



$2M_p = \frac{W_c L}{8}$

$W_c = \frac{16 M_p}{L}$

$M_p + \frac{1}{2}M_p = \frac{W_c L}{4}$

$W_c = \frac{6 M_p}{L}$

using analytical approach

$M_p = 0.686\left(\frac{W_c L}{8}\right)$

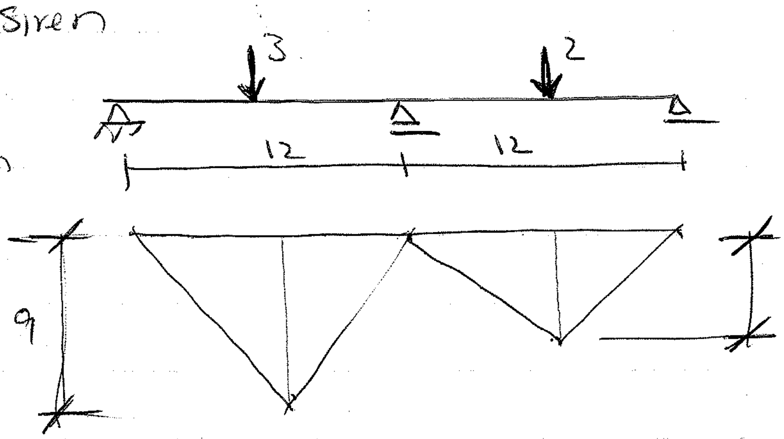
$W_c = \frac{8 M_p}{0.686 L}$

For the time being only contrated loads will be considered to avoid the complication until next chapter.

~~DESIGN PROBLEM~~

EXAMPLE: We have a continuous beam of constant cross section, under the given applied loads.

Consider a Mechanism as shown. Then



$M_p + \frac{M_p}{2} = 9$

$M_p = 6$

This is a correct solution, since

Equil. by B.M.D.

Mechanism is formed

Yield condition is satisfied

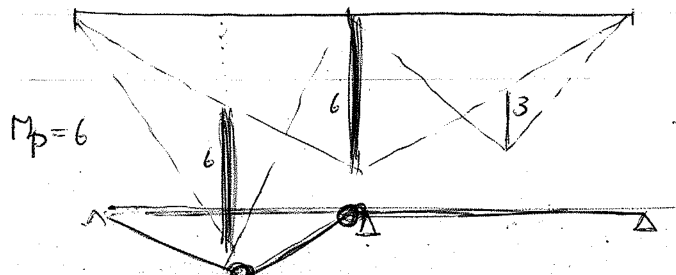


Fig. 2.

(A)

$M_p + \frac{1}{2} M_p = 6$

$M_p = 4$

incorrect solution because Yield condition is not satisfied unsafe solution

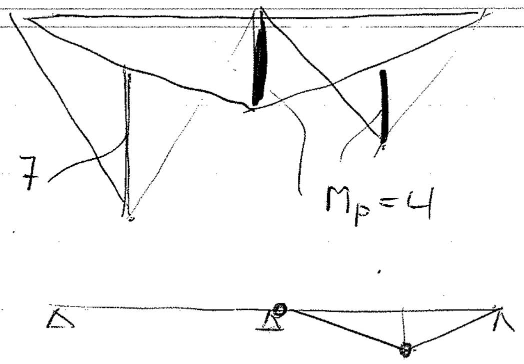


Fig. 2.

This unsafe property of solution corresponding to an incorrect mechanism of collapse is general, however, the statement can be made that

> Any solution derived from an arbitrary mechanism is unsafe, or at best correct (safe) if the mechanism happens to be correct one.

This indicates a possible approach for problem of plastic design of complex frames. For continuous beams this approach is convenient, however, for complex frames because of existing very many possible collapse mechanisms it is not workable.

Another application of the above unsafe solution is

$$7 \geq M_p \geq 4$$

this unsafe solution satisfies Equil. and Mechanism condition and as will be seen it provides a safe solution. i.e. if a beam of uniform cross section with $M_p = 7$ is designed it will carry the load safely.

In above example $(M_p)_{ext} = 6$ which falls in this range. These bounds are wide and for complex frames it will be made much narrower.

2.43 CONTINOUS BEAM WITH NONUNIFORM CROSS-SECTION

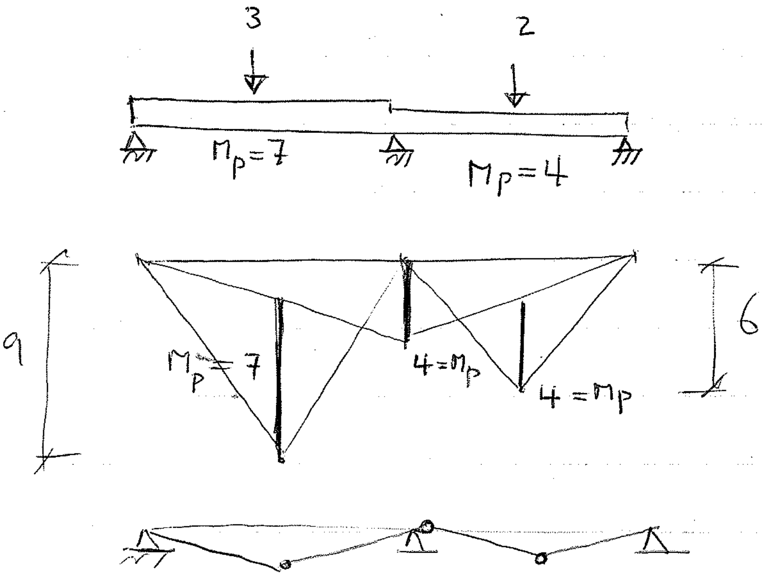
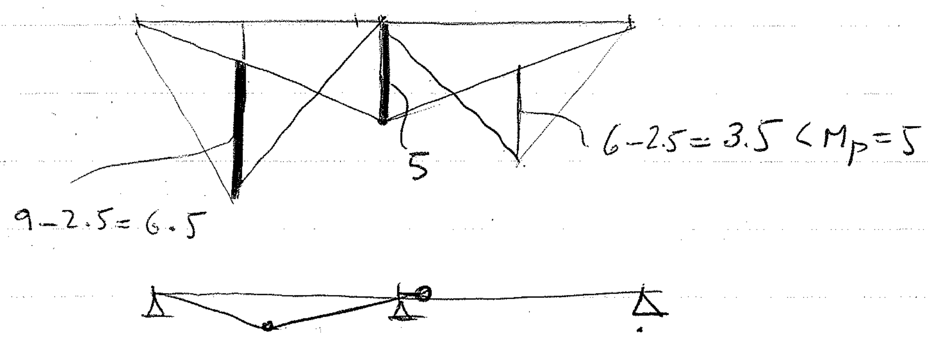


Fig. 2.
(B)

We start with $M_p=4$ for right hand span consider a mechanism as  and find moment of 7 at mid left span. If we take $M_p=7$ for left span, Then a good design is made, which produces the above mechanism.

NOW Take two spans with $M_p=6.5$ and $M_p=5$ The bending moment diagram is as below



Now Take weight of beam proportional to M_p Then we can compare The designs

Fig. 2.
(C)

	Design	Fig	M_p(left)	M_p(right)	Weight (arbitrary scale)
	1	2.A	6	6	12L
OPTIMAL →	2	2.B	7	4	11L
	3	2.C	6.5	5	11.5L
(See the Next Page) optimal →	4	D	4+cover plate	4	9.05L

2.4.4 ~~DESIGN WITH COVER PLATE~~

In order to avoid the expense of connecting two spans of different M_p s, one can use cover plate. As an example, one can use cover plate of length **4.21** for the left span as shown.

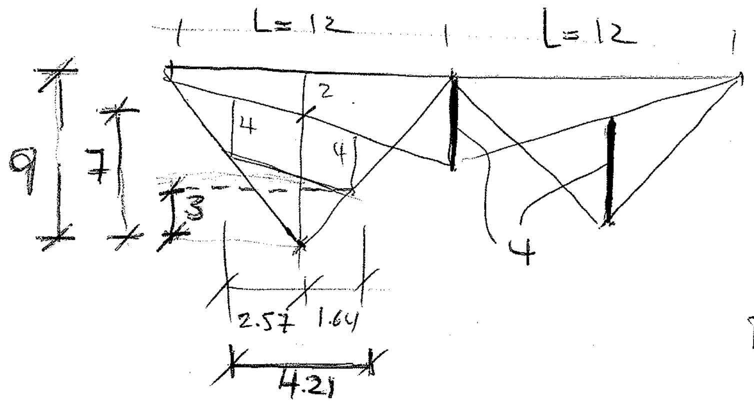


Fig. 2 use of cover plate
(D)

The reinforcement should provide an additional full plastic moment of 3 units, using the principles of chapter 1. The weight of such a reinforced design would be 9.05 L (compare with previous table).

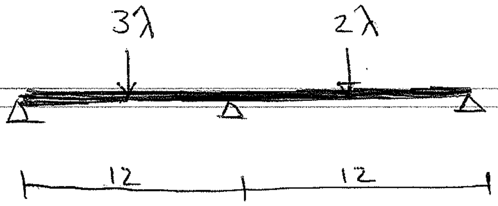
$$\frac{7.79}{12} L \times 4 + \frac{4.21}{12} L \times 7 + 4L = 9.05 L$$

ANALYSIS:

We studied design problem. Now we consider the analysis problem. Beam is uniform and both spans $M_p = 10$, and load factor λ is needed.

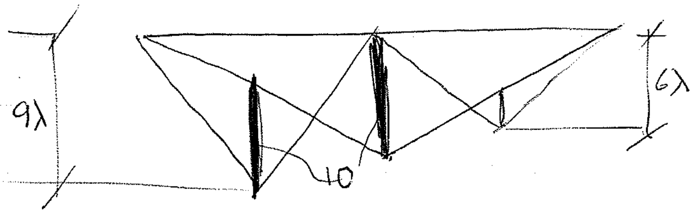
The two cases are studied in the next page. The simplicity of continuous beams is due to the simplicity of predicting the mechanisms.

EXAMPLE:

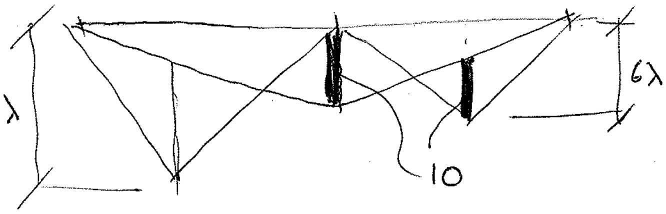


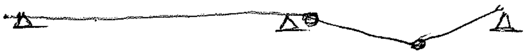
$M_p = 10$ for both spans

Find λ.



$9\lambda = 10 + 5 = 15$ $\boxed{\lambda = \frac{5}{3}}$



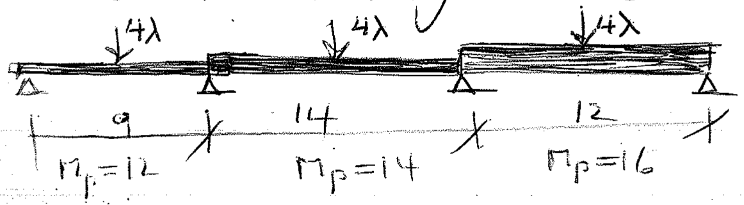


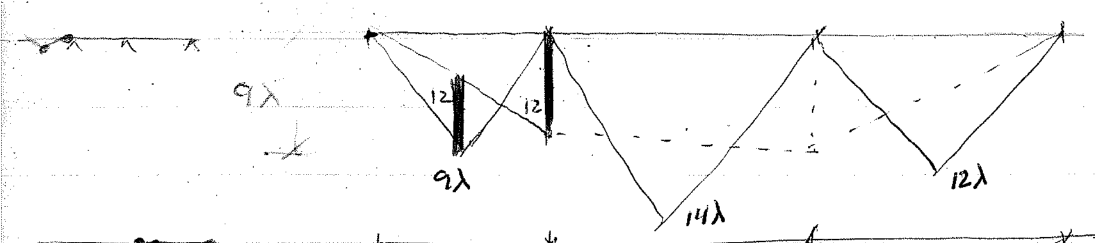
$6\lambda = 10 + 5 = 15$ $\boxed{\lambda = \frac{5}{2}}$

Thus $\lambda = \frac{5}{3}$ is the answer.

EXAMPLE:

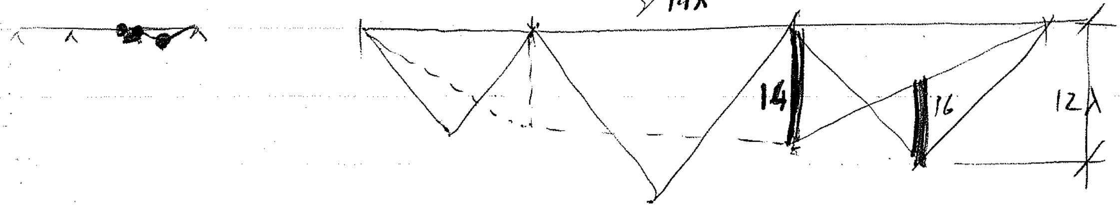
A 3 span beam is consider with three different Mp s as 12, 14 and 16. Loading is shown.





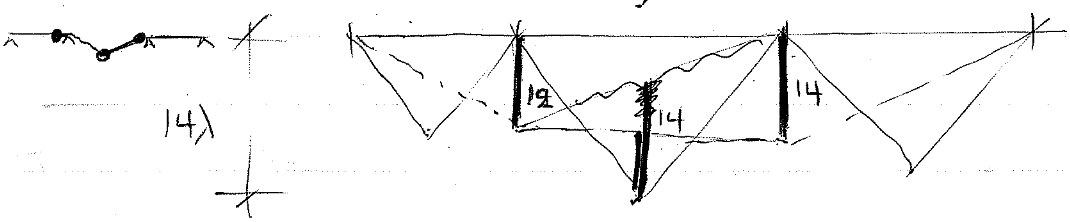
$9\lambda = 12 + 6 = 18$

$\boxed{\lambda = 2}$



$12\lambda = 23$

$\boxed{\lambda = 1.92}$



$14\lambda = 27$

$\boxed{\lambda = 1.93}$

And the smallest $\underline{\lambda = 1.92}$ is the answer.

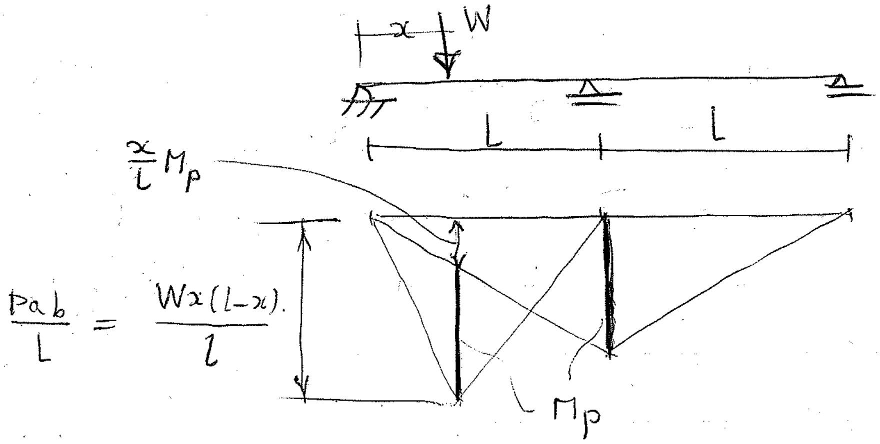
NOTE ! Hinge gets formed on weaker section at support.

NOTE ! Complete B.M. can not be constructed at this stage.

~~2.5. TRAVELLING LOADS~~ Moving Load

The concept of the influence line does not carry over to plastic analysis. It is possible to analyze a structure under the action of a single point load placed in an arbitrary position, but this information can not be used for the solution of a problem like Train. This is because the plastic solution involves non-linear behaviour, and principle of superposition does not hold.

EXAMPLE:



$$M_p + \frac{x}{L} M_p = \frac{Wx(L-x)}{L}$$

or
$$M_p = \frac{Wx(L-x)}{(L+x)}$$

The larges value happens for $\frac{dM_p}{dx} = 0$ leading to:

$$x(L-x) = (L-2x)(L+x)$$

or
$$x = (\sqrt{2}-1)L = 0.414\,L$$

Thus the worst position of load is slightly off-centre in one span and the corresponding M_p is

$$M_p = (3-2\sqrt{2})WL = 0.686\left(\frac{WL}{4}\right)$$

A wheel can be represented by two loads separtated by a small distance, representing the wheel of a crane.

2.6 THE WORK EQUATION

The sketching of B.M.Ds composed of free and reactant parts is an almost essential preliminary to the analysis or design of even quite a complex frame. Although this gives insight, however, analytical approaches are more convenient. Fundamental tool here is the equation of virtual work.

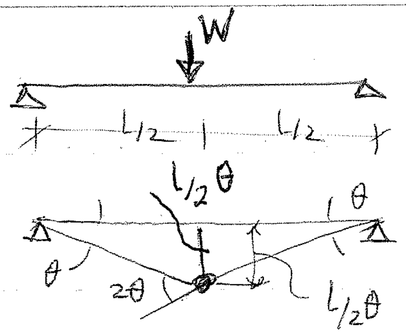
For the moment, it is convenient to regard the virtual work equation in the greatly restricted sense of a simple energy balance for the structure in collapse state.

Suppose a frame is on the point of collapse by the formation of plastic hinges, so that a small deformation of the collapse mechanism can occur at constant values of the applied loads. During such a small deformation, a typical load W will do work by moving through a certain distance, say δ, the magnitude of the work done is simply $\Sigma W\delta$, where the summation extends over all the loads on the frame.

This work done by the external loads will be absorbed in the rotating plastic hinge, which turn through certain angles, say θ, at constant bending moment M_p. The work dissipated in the hinges is therefore simply $\Sigma M_p\theta$, so that

$$\boxed{\Sigma W\delta = \Sigma M_p\theta}$$

SIMPLE EXAMPLE:



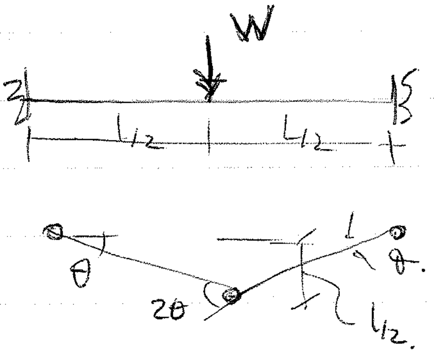
W. EQ. $\frac{L}{2}\theta W = M_p(2\theta)$

$$M_p = \frac{WL}{4}$$

θ is cancelled. This is always possible for mechanism of one degree of freedom.

The apparent restriction implied by ignoring the elastic compared with the plastic deflexions disappears if The equation of virtual work is used in place of Eq. of work (will be discussed later). However, it is probably best for the time being to regard equation $\Sigma W\delta = \Sigma M_p\theta$ as a "real" work equation in which elastic terms are so small as to be negligible; all collapse mechanisms will be drawn with straight lines between hinge points.

EXAMPLE 2.



W. EQ.

$$W\left(\tfrac{1}{2}L\theta\right) = (M_p)(\theta) + (M_p)(2\theta) + (M_p)(\theta)$$

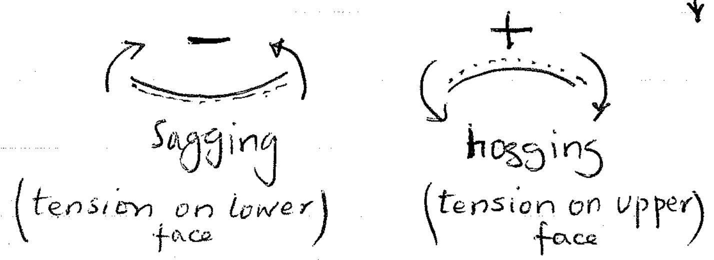
$$M_p = \frac{WL}{8}$$

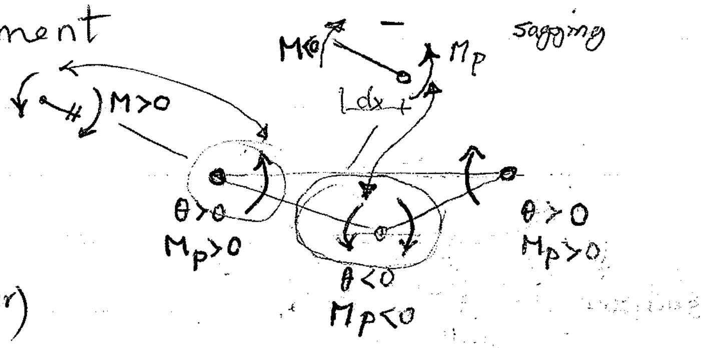
Since plastic work dissipated at a hinge is always positive, the numerical values of rotations could be written as 4θ, and total work dissipated being $(M_p)(4\theta)$.

2.6.1 ~~SIGN CONVENTION~~

A consistant sign convention is needed for the full application of the virtual work, and for other computational methods described later.

Hogging bending moments will be denoted positive, so that, for the fixed-ended beam of the following figure, the plastic hinge moments acting at the ends of the beam will be positive. The sagging central hinge will form under a negative bending moment

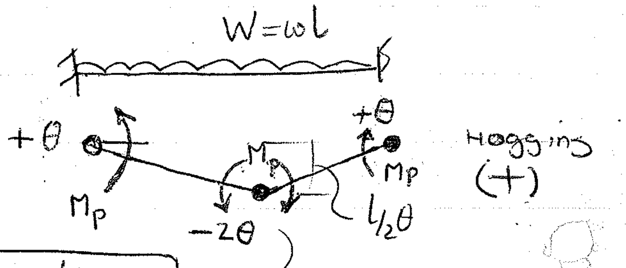




NOTE: For present purpose, whatever sign convention is used, The sign of hinge rotation must be taken to the same as the sign of the corresponding full plastic moments. Thus in work equation a positive hinge rotation will be multiplied by a positive value of M_p and negative rotation by a negative M_p, and the product is always positive.

2.6.2 FURTHER EXAMPLES:

EXAMPLE 3

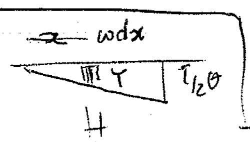


The uniform load W moves through an average distance $\frac{1}{4}l\theta$, so

$$W\left(\tfrac{1}{4}l\theta\right) = M_p(4\theta)$$

$$M_p = \frac{WL}{16}$$

$$W_c = \frac{16M_p}{l}$$



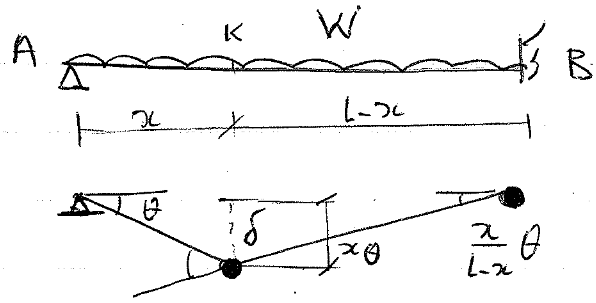
$W = wL$

$\frac{x}{L/2} = \frac{y}{L/2\,\theta}$

$y = \theta x$

$\text{Work} = 2\left[\int_0^{L/2} w y\,dx\right] = 2w\theta\left[\frac{x^2}{2}\right]_0^{L/2} = \frac{w\theta L^2}{4} = \frac{WL\theta}{4}$

EXAMPLE 4



$W = wl$

$$\theta + \frac{x}{L-x}\theta = \frac{L}{L-x}\theta$$

Location of sagging plastic hinge is denoted by x. If θ is the rotation of A, Then $\delta = x\theta$ and $\frac{x}{L-x}\theta$ is the rotation at B. Rotation at K is sum of θ at A and B i.e. $\frac{L}{L-x}\theta$

Suppose the uniform load W moves through an average distance $\frac{1}{2}x\theta$, Then Work EQ. is

$$W\left(\frac{1}{2}x\theta\right) = M_p\left[\frac{L}{L-x}\theta + \frac{x}{L-x}\theta\right]$$

or
$$M_p = \frac{Wx(L-x)}{2(L+x)}$$

The unsafe Theorem can now be used to find x. The value of x will be correct which leads to greatest value of M_p (i.e. the safe design).

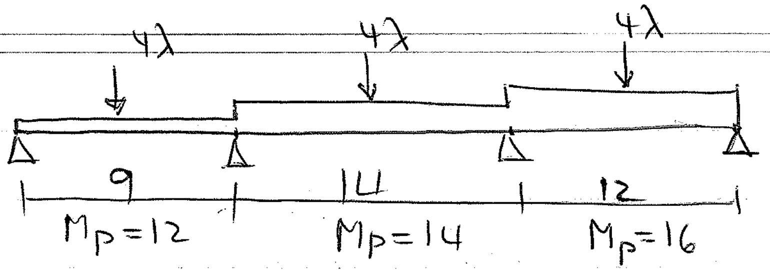
$$\frac{dM_p}{dx} = 0 \Rightarrow x = (\sqrt{2}-1)L$$

$$x(l-x) = (l-2x)(L+x) \Rightarrow x^2 + 2xl - l^2 = 0 \quad x = \frac{-l \pm \sqrt{l^2+l^2}}{1}$$

Substituting back

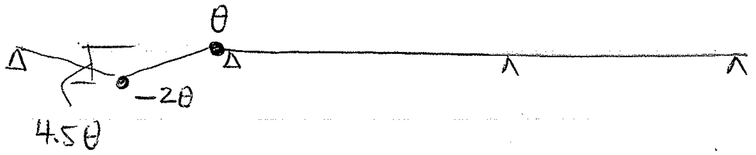
$$M_p = \left(\frac{3}{2} - \sqrt{2}\right)Wl = 0.686\,(Wl/8)$$

EXAMPLE 5



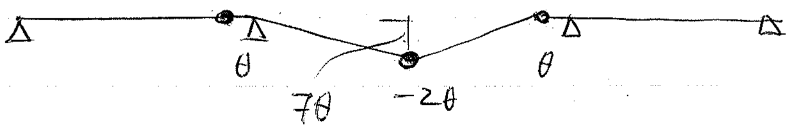
$4\lambda(4.5\theta) = (12)(3\theta)$

$\lambda = 2.00$



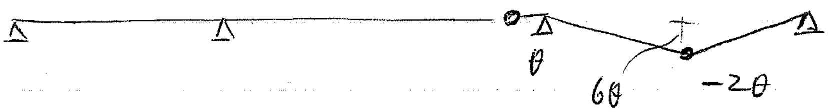
$4\lambda(7\theta) = (12)(\theta) + (14)(3\theta)$

$\lambda = 1.93$



$4\lambda(6\theta) = (14)(\theta) + (16)(2\theta)$

$\boxed{\lambda = 1.92}$



This is the same result as before, showing that the right hand span is critical.

Graphical methods are much informative also analytical approach, so we will continue with graphical method, for the time being.

2.7 RECTANGULAR PORTAL FRAMES

2.7.1 Assumptions :

a) external loads are resisted by bending of the members

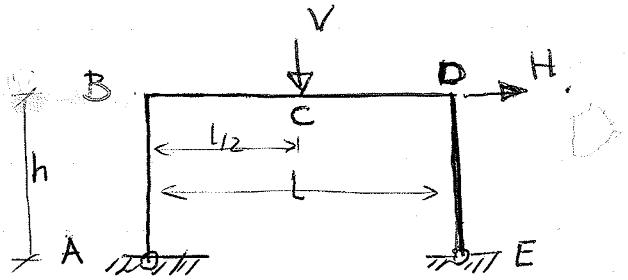
b) shear and axial loads cause only secondary effects, i.e. also M_p s can be adjusted, however, no individual member becomes unstable

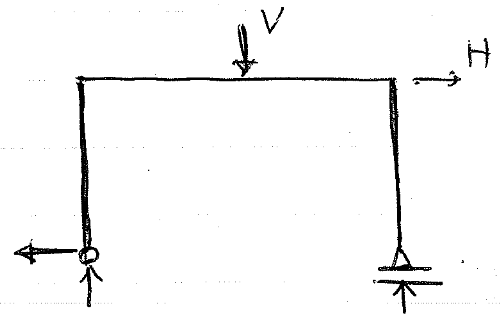
c) In all our disscusions we assume that each member, and the whole structure, is in stable equilibrium right up to the collapse state.

2.7.2 PORTAL FRAMES WITH PINNED FEET

consider a rectangular frame of uniform section with pinned feet, which has one redundancy

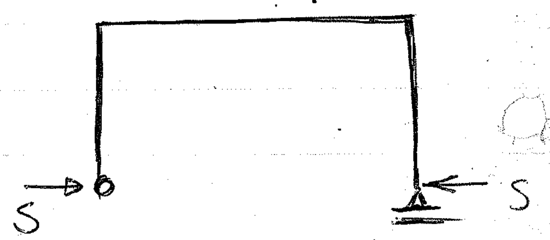
Two highly idealised loads are considered; V: dead load + superimposed roof load and H: wind load.





a) statically determinate (primary) structure

b) redundant force S



~~Bending moments are drawn~~

Free bending moment for Statically deter. Str. and Reactant B.M. for redundant S

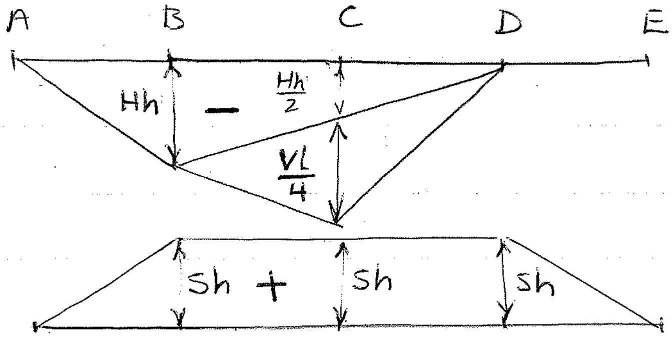
The frame has been "opened up" on to a horizontal base line for the purpose of plotting.

27.3 SIGN CONVENTION:

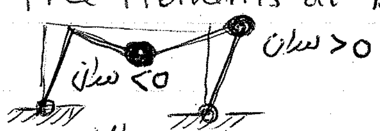
Hogging bending moments for the beam are denoted positive, as usual, and plotted above the base line, and the same convention is carried into the columns. So that a bending moment producing compression on the inside of the frame is positive.

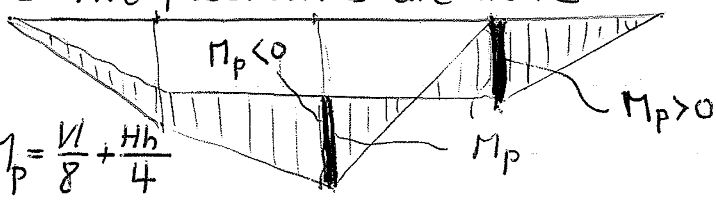
Free B.M.D. (−)

Reactant B.M.D (+)

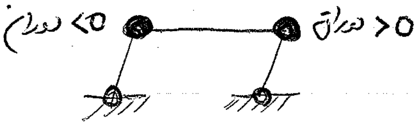


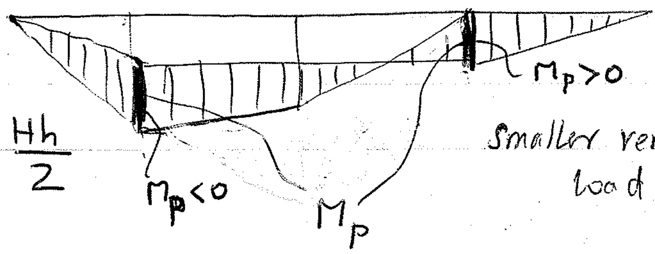
These diagrams are superimposed in such a way that a collapse mechanism is formed with TWO plastic hinges. Depending on Free Moments at B and C two possibities are there





$Sh = M_p$ & $\frac{VL}{4} + \frac{Hh}{2} - Sh = M_p \Rightarrow M_p = \frac{Vl}{8} + \frac{Hh}{4}$

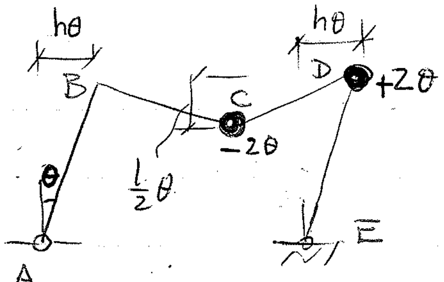




$Sh = M_p$ & $Hh - Sh = M_p \Rightarrow M_p = \frac{Hh}{2}$

Smaller vertical load

Using work equation one can also obtaine the same equation. e.g. for the first case



$$(H)(h\theta)+(V)\left(\tfrac{1}{2}L\theta\right) = M_p(4\theta)$$

i.e $$M_p = \frac{VL}{8} + \frac{Hh}{4}$$

C, D plastic hinges

A, E normal hinges

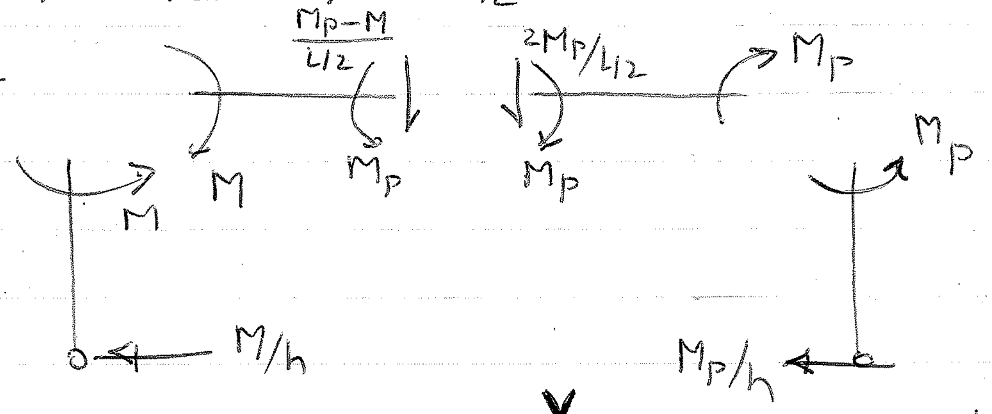
2.7.4 FURTHER INVESTIGATION OF PINNED PORTAL FRAMES

We study the collapse mode shown above, and we draw the free body diagram.

1) put M_p at C and D considering the mode of collapse (sagging in C and hogging in D)
2) At B put M which is unknow at this stage
3) Obtain shear in E (M_p/h)
4) obtain shear in C $\left(\frac{2M_p}{L/2}, \frac{M_p-M}{L/2}\right)$
5) obtain shear in D (M/h)
6) write total shear at cols feet equal to H $\left(H=\frac{M}{h}+\frac{M_p}{h}\right)$
7) write ΣF_Y for C $\left(\frac{M_p-M}{L/2}\right)+\left(\frac{2M_p}{L/2}\right)=V$
8) solve to find M and M_p

$$\begin{cases} M+M_p = Hh \\ -M+3M_p = \frac{VL}{2} \end{cases}$$

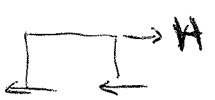
$$\boxed{\begin{aligned} M_p &= \frac{Hh}{4}+\frac{VL}{8} \\ M &= \frac{3Hh}{4}-\frac{VL}{8} \end{aligned}}$$



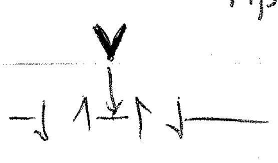
⇓

$M+M_p=Hh$

$M=Hh-M_p$



$\Sigma F_x=0$



$\Sigma F_Y=0$

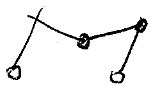
NOTE: M_p has the same sign of θ and its direction is such that opposes the rotation.

The Solution is a correct one if Yield Condition is also satisfied. i.e $-M_p \leq M \leq M_p$ or

$$-M_p \leq Hh - M_p \leq M_p$$

$$\boxed{0 \leq Hh \leq 2M_p}$$

Using $M_p = \frac{Hh}{4} + \frac{VL}{8}$ we get

$$\boxed{VL \geq 2Hh \geq 0}$$ Condition for Mech to happen 

For $VL \leq 2Hh$ pure sidesway will occur.

2.7.5 ANALYTICAL METHOD

Bending moment at each section is the sum of Free B.M.D and Reactant B.M.D. Thus at B, C and D

(See Fig in page 24) $\Rightarrow$

$$M_B = -Hh + Sh$$
$$M_C = -\left(\frac{Hh}{2} + \frac{VL}{4}\right) + Sh$$
$$M_D = Sh$$

(sagging moment is considered −)

If collapse happens with hinges at C and D, then the last two equations become

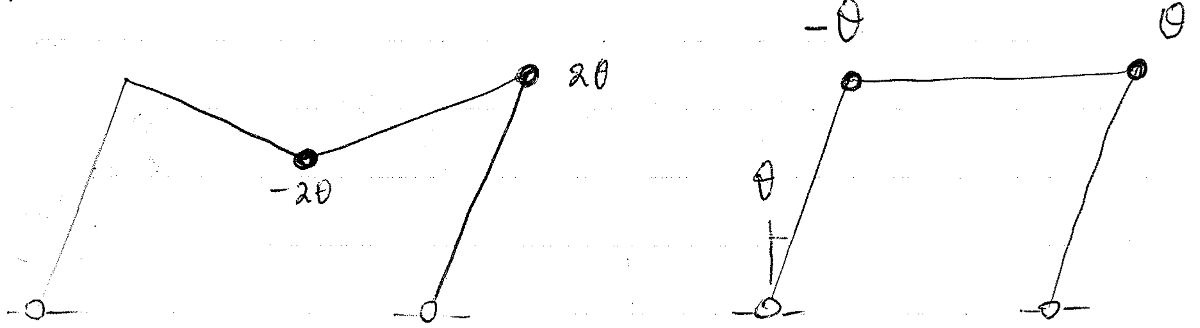
$$\left.\begin{aligned} C:\quad -M_p &= -\left(\frac{Hh}{2} + \frac{VL}{4}\right) + Sh \\ D:\quad M_p &= Sh \end{aligned}\right\} \rightarrow M_p = \frac{Hh}{4} + \frac{VL}{8}$$

The sign of B.M. at plastic hinges agree with the signs of hinges rotations. The unwanted redundant S may be eliminated, leaving a single collapse equation.

IN GENERAL if $\gamma(S) = R$, then $R+1$ equations can be written for $R+1$ Plastic hinges, so that all redundant can be found and one EQ. is left as a single collapse equation.

2.7.6 YIELD SURFACE and INTERACTION DIAGRAMS

consider the same portal as before with two collapse modes as reshown in the following, with corresponding collapse equations.



$$M_p = \left(\frac{Hh}{4}\right) + \left(\frac{Vl}{8}\right) \qquad\qquad M_p = Hh/2$$

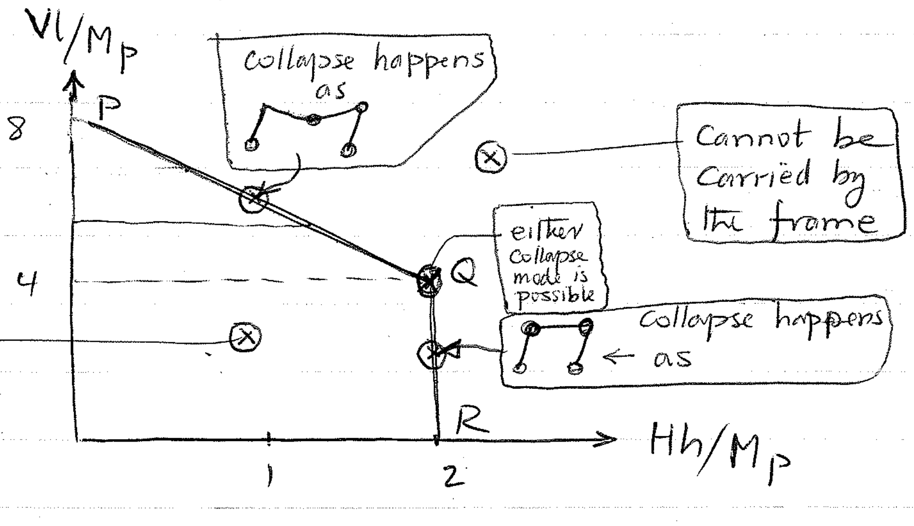
plot these with axes showing V and H. This diagram is another example of yield surface.

$$1 = \frac{1}{8}\left(\frac{Vl}{M_p}\right) + \frac{1}{4}\left(\frac{Hh}{M_p}\right)$$

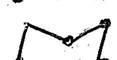
$$\left(\frac{Vl}{M_p}\right) = 8 - 2\left(\frac{Hh}{M_p}\right)$$

$= -2$, $= \frac{1}{2}$

certain possible combination of V and H for which structure is safe



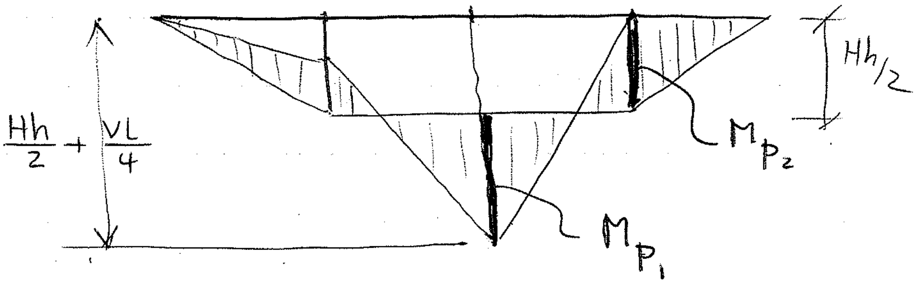
Yield Surface

property of yield surface: Orthogonality rule; i.e. if two axes of deformation (δ_H, δ_V) coincide with (H, V), the direction vertical to yield surface shows the relative directions of the points where loads are applied. e.g. vertical to QR is horizontal i.e $\delta_V = 0$ and δ_H exists. vertical to PQ is $\delta_V/\delta_H = l/2h$ corresponding to 

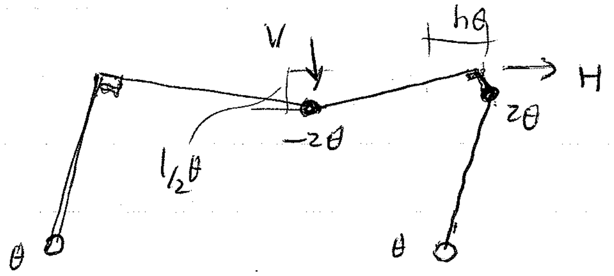
~~2.7.7 NON-UNIFORM PIN-BASED DESIGN~~

consider M_{P_2} for columns and M_{P_1} for beam and $M_{P_2} < M_{P_1}$.

From B.M.D. we have



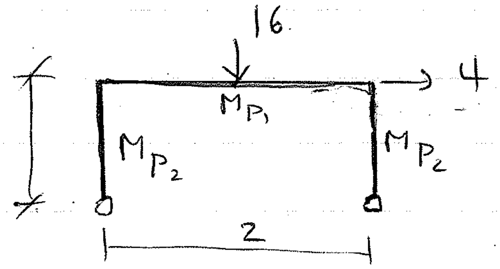
$$M_{P_1}+M_{P_2}=\left(\frac{Hh}{2}\right)+\left(\frac{VL}{4}\right)$$



There is a minimum value of $M_{P_2}\left(=\frac{Hh}{2}\right)$, but a range of values M_{P_1} and M_{P_2} Satisfying the above EQ. will give a design of the frame if the Free B.M.D is of the general shape sketched in Fig. above..

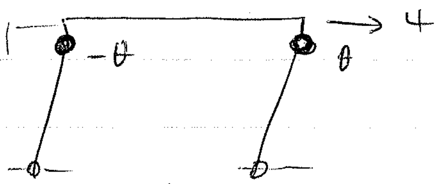
NUMERICAL EXAMPLE AND DESIGN PLANE

Consider a frame as shown



For differen Mechanisms we have:

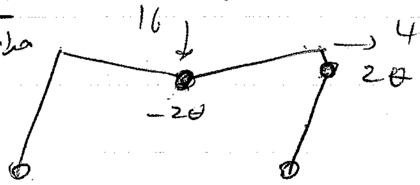
a) $(4)(\theta \times 1) = (M_{P_2})(2\theta)$



(1) $\boxed{M_{P_2}=2}$ Drawing B.M.D. one finds out that $M_{P_1} \geq 8$ (bending moment at center of beam)

$\frac{Hh}{2}+\frac{VL}{4} = \frac{4\times 1}{2}+\frac{16\times 2}{4}=10=2+M_{P_1}$ $M_{P_1}=8$ ملهر

b) If Beam Section is reduced below 8 Then → M_{P_2} should be ~~increased. Then The front mode~~ will happen

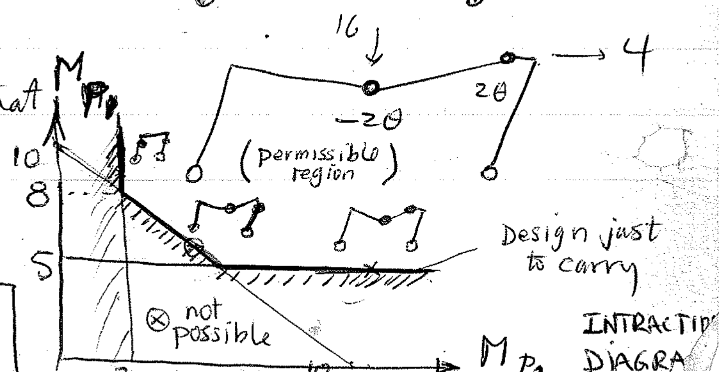


(۲) $\boxed{2M_{P_1}+2M_{P_2}=20}$

It is assumed that $M_{P_2} < M_{P_1}$

c) If M_{P_1} is further reduced so that $M_{P_1}=M_{P_2}$ Then $4M_{P_1}=20$

and $\boxed{M_{P_1}=5}$ (۳)



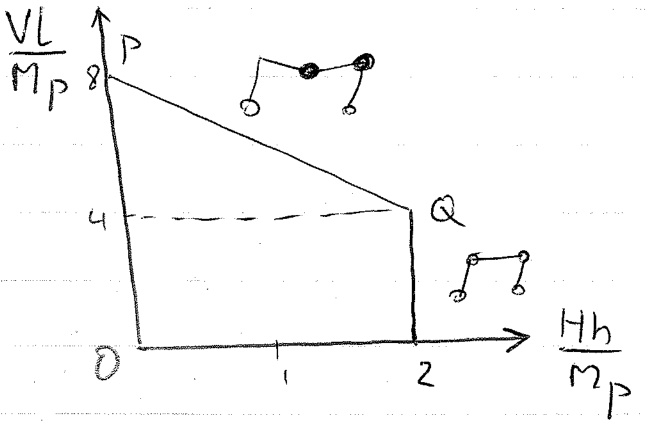
شکل بعد رسم معادلات (۱)، (۲)، (۳) است

DESIGN PLANE

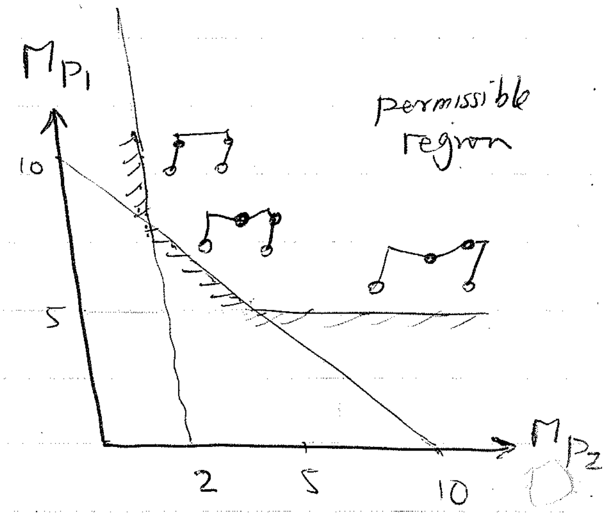
NOTE: The boundary of permissible region is in fact a kind of inverse of the yield surface given before.

Yield surface corresponds to a given structure which can be subjected to different combination of loading, while The boundary of the permissible region in the design plane corresponds to a given loading acting on structure whose design is varied.

Thus in Theory, the boundary of the permissible design could be examined to find the most economical frame to carry a given set of loads.



YIELD SURFACE
Design fixed-loading varied



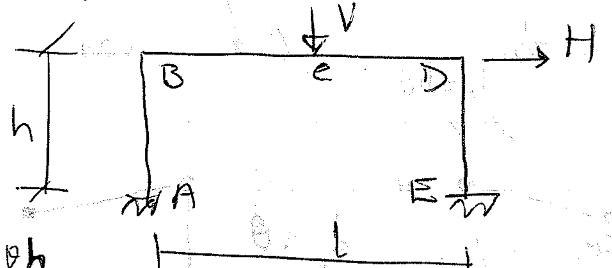
INTERACTION DIAGRAM
Load fixed - design varied

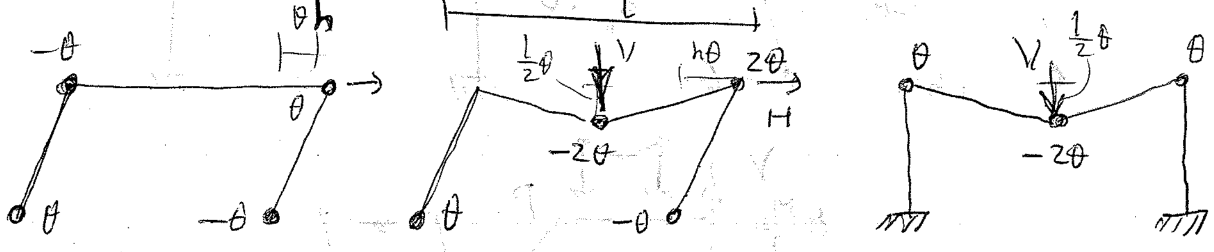
NOTE:

Yield surface around O can not have reentrant angle. Thus in an analysis if a collapse mode is forgotten, a reentrant angle may appear, concluding the incompleteness of the analysis.

2.7.8 THE FIXED-BASE RECTANGULAR PORTAL FRAME

For frame with uniform Cross-section shown in Fig. 2.
Three collapse modes are possible as shown





(a) High value of horizontal load.

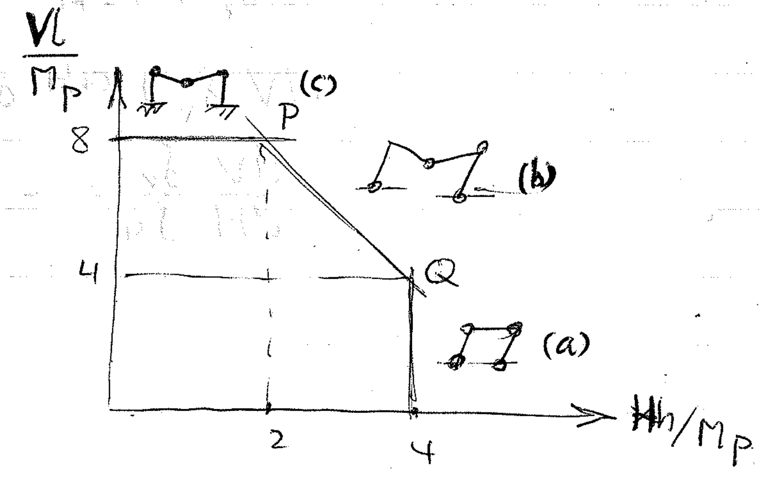
(b)

(c) High value of vertical load.

The collapse equations are as

$$
\begin{aligned}
&a) \quad (H)(h\theta) = (4\theta)M_p \\
&b) \quad (\tfrac{L}{2}\theta)(V) + (h\theta)(H) = (6\theta)M_p \\
&c) \quad (\tfrac{L}{2}\theta)(V) = (4\theta)M_p
\end{aligned}
\qquad
\left\{
\begin{aligned}
Hh &= 4M_p \\
Hh + \tfrac{1}{2}VL &= 6M_p \\
\tfrac{1}{2}VL &= 4M_p
\end{aligned}
\right.
$$

The Yield surface ~~interaction~~ diagram is shown in Fig. 2.



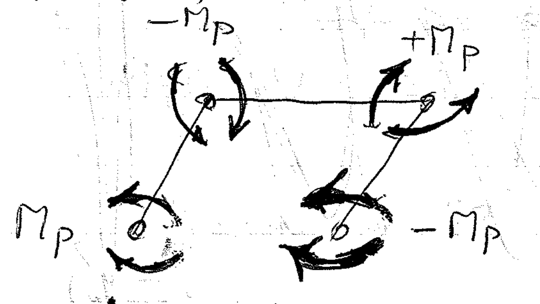
Yield surface

For no vertical load (a) is a correct mechanism

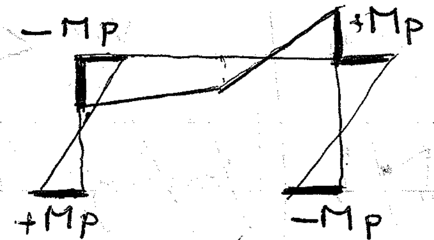
For no horizontal load (c) is a correct mechanism

Let us construct the bending moment diagrams

For mode (a) we have

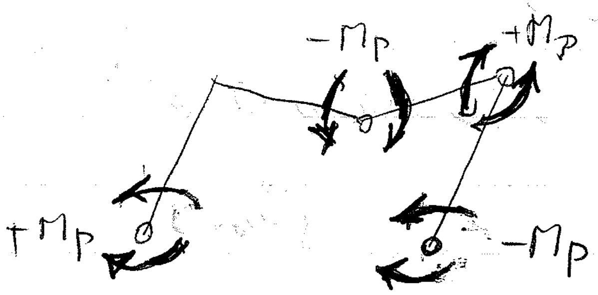


(a)

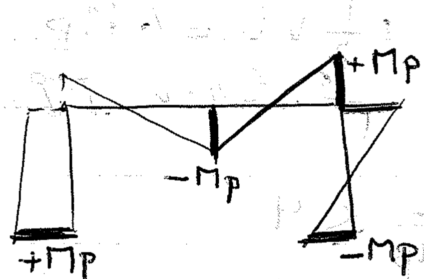


B.M.D. for (a)

For mode (b) we have

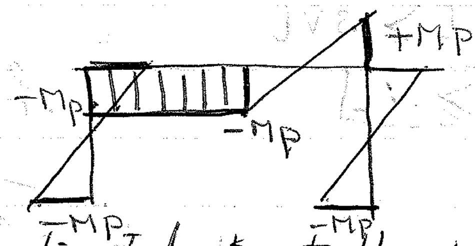


(b)



B.M.D. for (b)

For point Q mode (a) and (b) should happen simultaneously i.e



Half of the beam is subjected to full plastic moment

Thus limits may be assigned to the values of V and H for collapse by various modes:

a) $Hh = 4M_p \quad (Hh \geq Vl)$

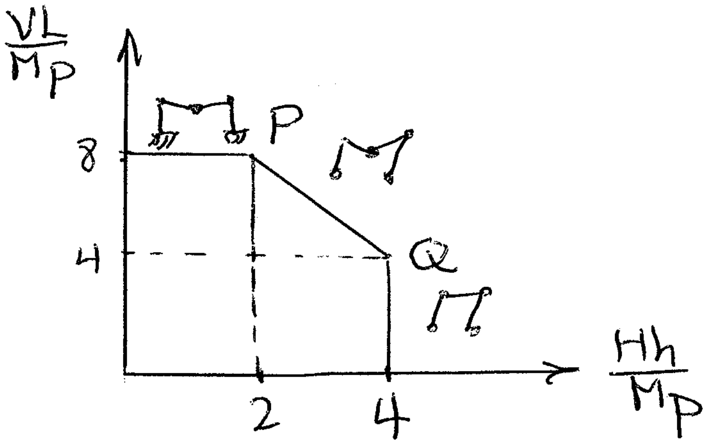
b) $Hh + \frac{1}{2}Vl = 6M_p \quad (Vl \geq Hh \geq \frac{1}{4}Vl \text{ or } Hh \leq Vl \leq 4Hh)$

c) $\frac{1}{2}Vl = 4M_p \quad (Vl \geq 4Hh)$

Look at the interaction Diagram (Exercise for students) plot $-M_p \leq M \leq M_p$

The same as pages 25&26 static analysis can be made

Conditions for different modes



PROOF:

Consider the equation of PQ as

$$Hh + \frac{1}{2}VL = 6M_p \qquad (1)$$

که Q مربوط به آنست. برای مود باید داشته باشیم

$$\frac{VL}{M_p} \leq 4$$ از روی شکل

$$M_p \geq \frac{VL}{4} \qquad (۲)$$

$$\frac{1}{6}\left(Hh + \frac{1}{2}VL\right) \geq \frac{VL}{4}$$ با تلفیق (۱) و (۲) داریم

i.e $\quad 2Hh + VL \geq 3VL$

i.e $\quad \boxed{Hh \geq VL}$ حالت اول اثبات شد.

برای مود باید داشته باشیم. از روی شکل برای نقاط واقع بین A و B

$$\frac{Hh}{M_p} \leq 2 \qquad \text{i.e.} \quad M_p \geq \frac{Hh}{2}$$

با قرار دادن در معادله PQ یعنی (۱)

$$\frac{1}{6}\left(Hh + \frac{1}{2}VL\right) \geq \frac{Hh}{2}$$

i.e $\quad Hh + \frac{1}{2}VL \geq 3Hh$

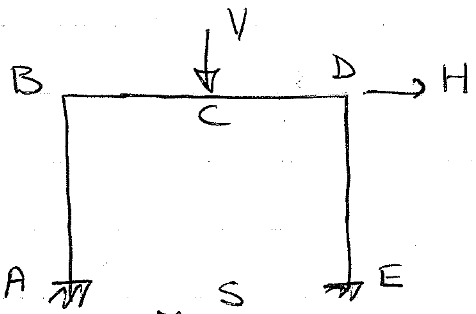
$$\boxed{VL \geq 4Hh}$$ حالت سوم ثابت شد.

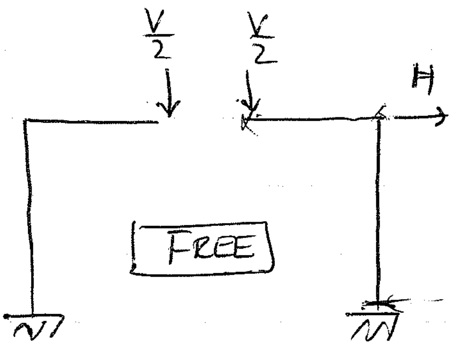
برای حالت دوم: برای نقاط واقع بین C و D باید داشته باشیم $4 \geq \frac{Hh}{M_p} \geq 2$ و یا معادل آن

از $8 \geq \frac{VL}{M_p} \geq 4$ استفاده کرد. با تلفیق هرکدام با (۱) ثابت می‌شود که این روابط حاکم است

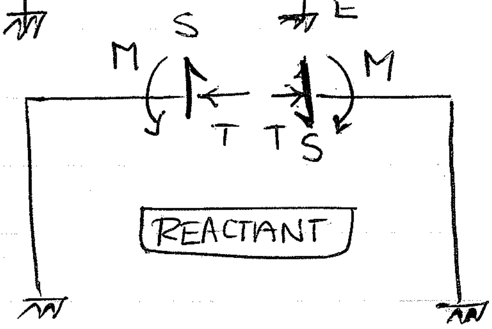
~~2.7.9 DIFFERENT APPROACH FOR THE~~ ANALYSIS OF FIXED – BASE PORTAL FRAMES

This can be made be an examination of free and reactant bending moment diagrams



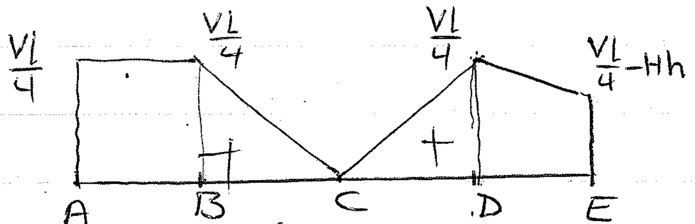


FREE

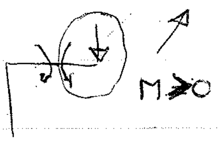


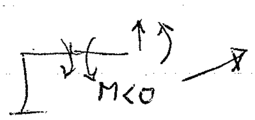
REACTANT

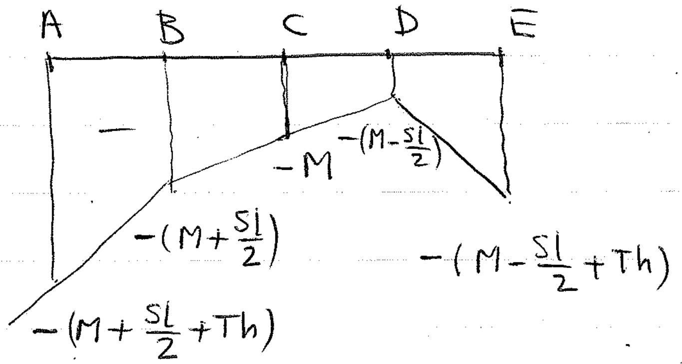
V is cut to $\frac{V}{2}$ and $\frac{V}{2}$ to keep the symmetry.



Free



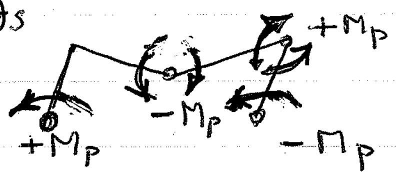




Reactant

For sign pay attention to the sign of θs

NOW Taking as an example mode (b)



mode (b)

$$\begin{cases} A: & \frac{VL}{4} - (M+\frac{SL}{2}+Th) = +M_p \\ C: & 0 - M = -M_p \\ D: & \frac{VL}{4} - (M-\frac{SL}{2}) = +M_p \\ E: & (\frac{VL}{4}-Hh) - (M-\frac{SL}{2}+Th) = -M_p \end{cases} \quad (1)$$

~~Eliminating T, M, S we obtain collapse equation as~~

$$\boxed{\frac{VL}{2} + Hh = 6M_p}$$

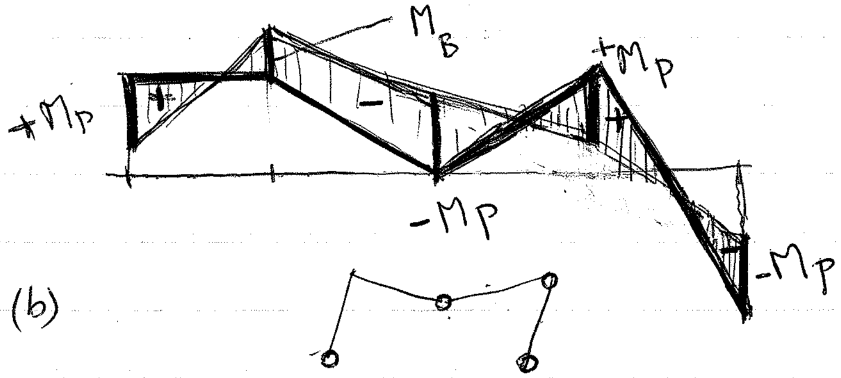
and back substitution yields

$$M = M_p = \frac{1}{12} VL + \frac{1}{6} Hh$$

$$\frac{1}{2} SL = -\frac{1}{12} VL + \frac{1}{3} Hh \qquad (*)$$

$$Th = \frac{1}{6} VL - \frac{2}{3} Hh$$

Using these values of redundants, the bending moment for mode (b) can be constructed completely



mode (b)

Using this M_B can be calculated to check for satisfying the yield condition

$$M_B = \left(\frac{VL}{4}\right) - \left(M + \frac{SL}{2}\right) = \frac{VL}{4} - \frac{1}{2} Hh$$

Substituting value of $M + \frac{SL}{2}$ from above equations (*)

M_B must satisfy $+M_p \geq M_B \geq -M_p$

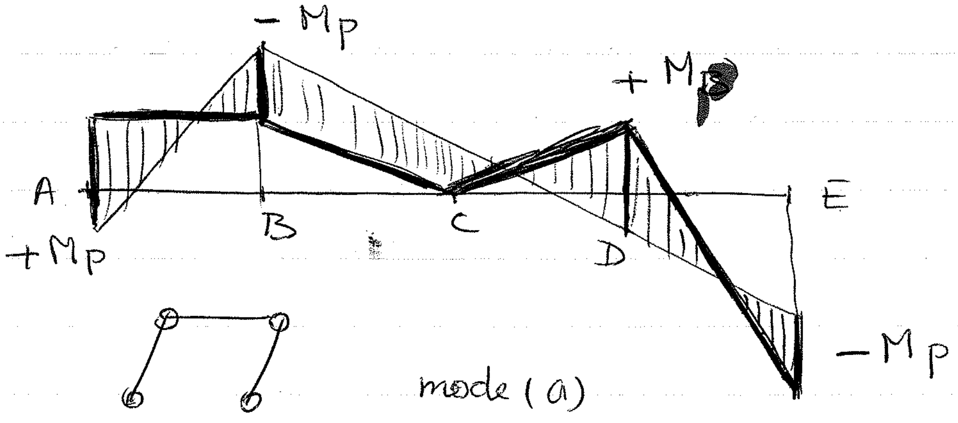
$$\frac{VL}{12} + \frac{Hh}{6} \geq \frac{VL}{4} - \frac{Hh}{2} \geq -\frac{1}{12} VL - \frac{1}{6} Hh$$

rearranging we get

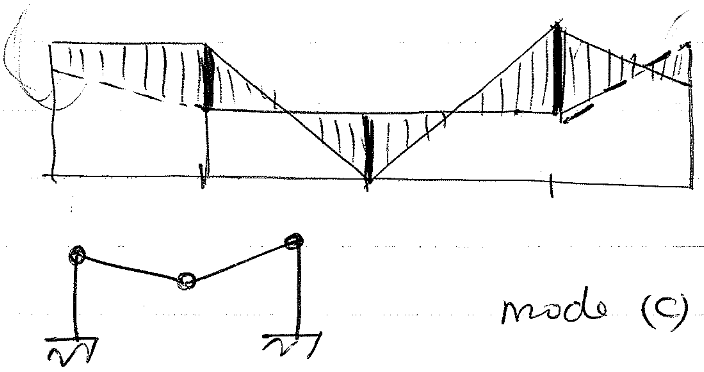
$$4Hh \geq VL \geq Hh$$

This is the same condition as found before.

For other modes



mode (a)



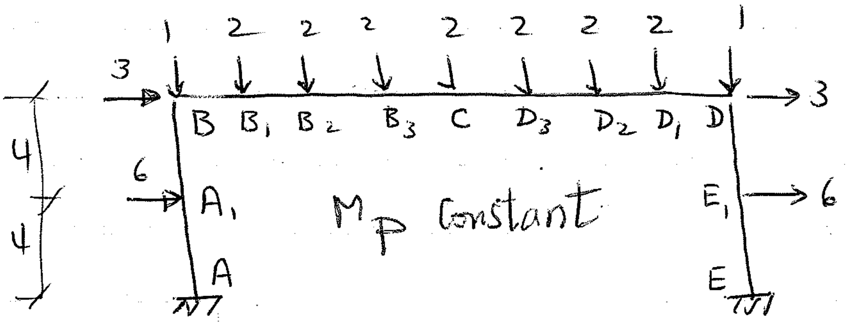
mode (c)

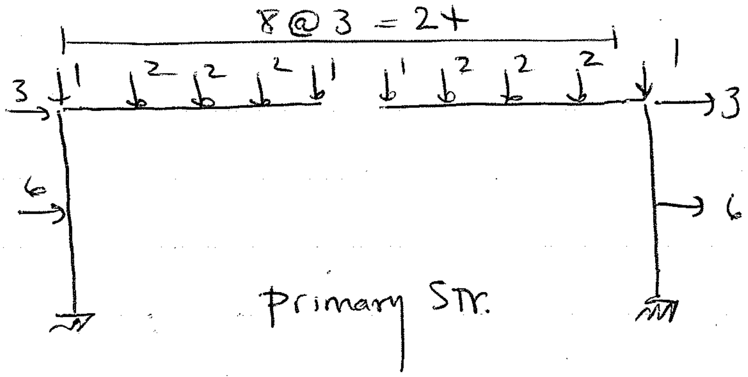
As noted above, the main virtue of writing equations (1) is that it gives a relatively easy way of computing collapse equations for a real frame acted upon by complex loading systems. The reactant moments for a fixed-base frame are always given by expressions of figures ~~above~~, independently of the loading on the frame; reactant moments are purely functions of the frame geometry. Thus if a table of free B.Ms is available for any given loading system, however complex, it is a simple matter to solve a set of equations similar to EQ.(1)

As An example we solve the following:

27.10 AN EXAMPLE WITH COMPLEX LOADING

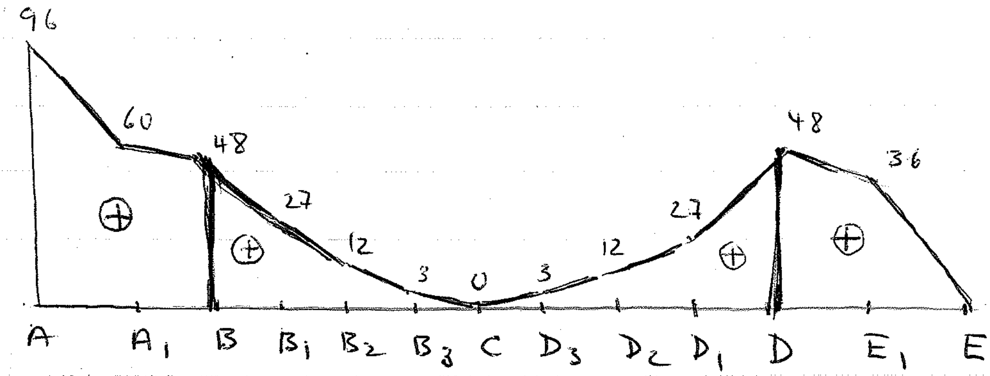
Design Problem:
Consider this frame, cut it at middle to obtain primary structure as shown below and calculate Free B.Ms as in Table sketched below



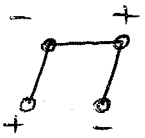


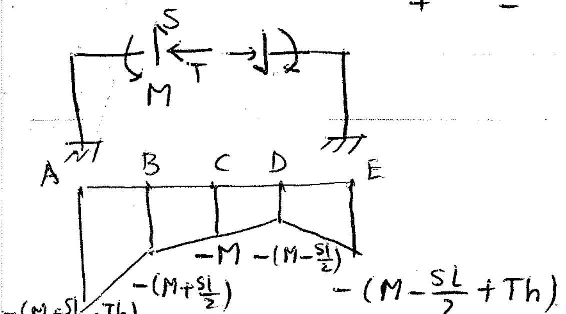
	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0

Free B.M.



suppose mode (a) is a correct mode with hinges at A, B, D and E. Consider resultant moments from





$$A: \quad 96-(M+12S+8T)=M_p$$
$$B: \quad 48-(M+12S)=-M_p$$
$$D: \quad 48-(M-12S)=M_p$$
$$E: \quad 0-(M-12S+8T)=-M_p$$

Solving these leads to

$M_p = 24$

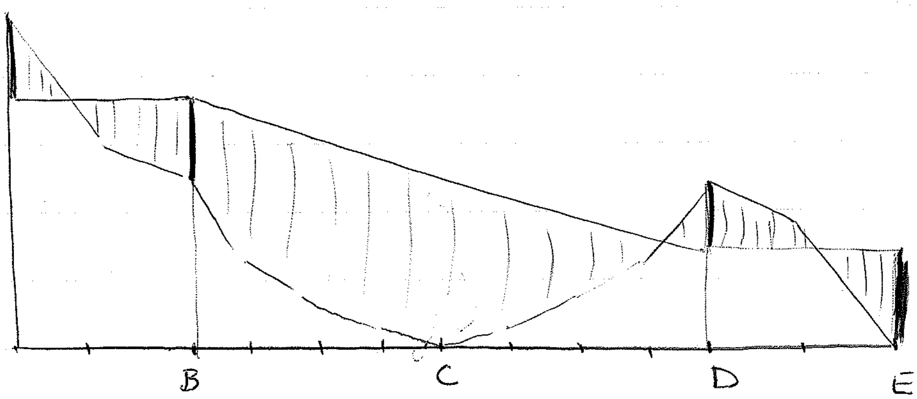
$M = 48$

$12S = 24$

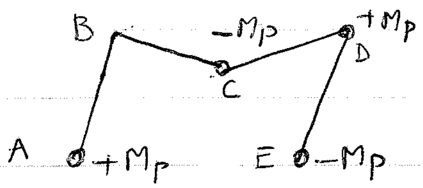
$8T = 0$

~~Since the reactant diagram consists of straight lines,~~
the full table of moments may be completed as

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
FREE	96	60	48	27	12	3	0	3	12	27	48	36	0
REACTANT	−72	−72	−72	−66	−60	−54	−48	−42	−36	−30	−24	−24	−24
Total	<u>24</u>	−12	<u>−24</u>	−39	−48	$\boxed{-51}$	−48	−39	−24	−3	<u>24</u>	12	<u>24</u>



It can be seen that the guess was wrong and yield condition is highly violated. From above figure it would seem that collapse mode (b) in much more nearly correct, and will now be tried as a second guess:



mode (b)

A: $96-(M+12S+8T) = M_p$

C: $0-(M) = -M_p$

D: $48-(M-12S) = M_p$

E: $(0)-(M-12S+8T) = -M_p$

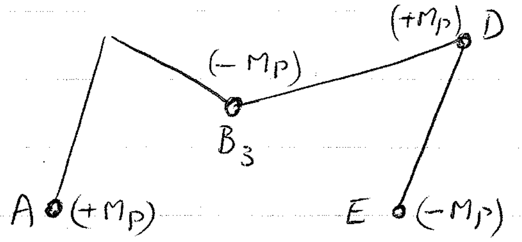
Solution is given as

$$M_p = M = 32$$
$$12S = 16$$
$$8T = 16$$

The total B.M. is given in the next Table.

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0
Reactant	−64	−56	−48	−44	−40	−36	−32	−28	−24	−20	−16	−24	−32
Total	32	4	0	−17	−28	$\boxed{-33}$	−32	−25	−12	7	32	12	−32

Collapse mode is not yet quite correct because B.M. at B_3 is more than M_p. It is almost certain that collapse mode is as shown below, where plastic hinge is formed at B_3 rather than C (due to the distributed nature of vertical loading), and this is confirmed as follows:



$$A: \quad 96 - (M + 12S + 8T) = M_p$$
$$B_3: \quad 3 - (M + 3S) = -M_p$$
$$D: \quad 48 - (M - 12S) = M_p$$
$$E: \quad 0 - (M - 12S + 8T) = -M_p$$

Solving these equations

$$M_p = 32.3$$
$$M = 31.4$$

and the final table is as below

$$12S = 15.7$$
$$8T = 16.6$$

	A	A_1	B	B_1	B_2	B_3	C	D	D_2	D_1
Free	96	60	48	27	12	3	0	3	12	27
Reactant	−63.7	−55.4	−47.1	−43.2	−39.2	−35.2	−31.4	−27.5	−23.5	−19.6
Total	32.3	4.6	0.9	−16.2	−27.2	−32.3	−31.4	−24.5	−11.5	7.4

	D	E_1	E
Free	48	36	0
Reactant	−15.7	−24.	−32.3
Total	32.3	12.0	32.3

Yield condition is satisfied. Thus it is the correct solution.

2.7.11 APPLICATION OF BAD GUESS IN DESIGN

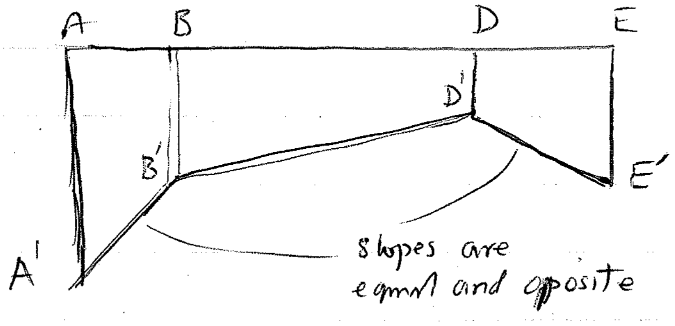
$M_p = 24$ was a bad estimate in the analysis we carried out for mode (a), the largest value of B.M. was $M_{B_3} = 51$. Thus we have bounds as

$$51 \geq M_p \geq 24$$

The corresponding Table would serve as a basis for the design of nonuniform frame. The columns could be given their minimum Full plastic moment of 24 unit and the beam 51 unit.

2.8 TRIAL AND ERROR GRAPHIC METHOD FOR DESIGN

The above design example probably best solved on a drawing board. On a plot of the free bending moments, must be superimposed by trial and error, a reactant line of the following form



This consists of 3 straight lines and slopes of the two columns are equal and opposite (since the slope of shear forces are the same in both columns for reactant forces).

The above figure requires the knowledge of 3 quantities to be specified. If the reactent moments

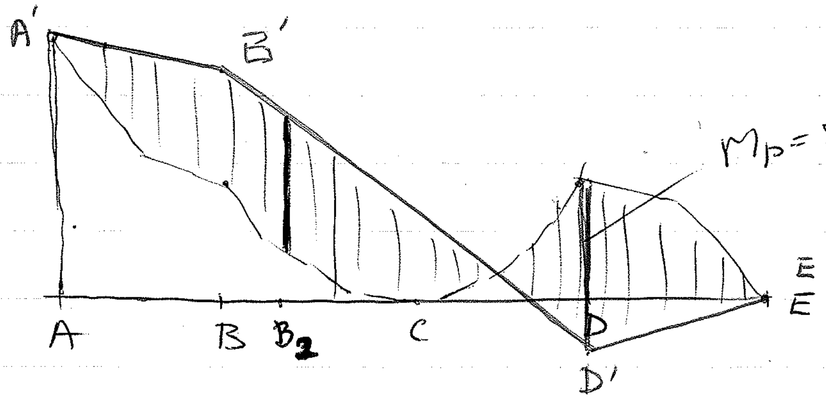
~~at D and E are known, for example, then line D'E' can~~ be drawn. If in addition, reactant moment at A is specified, then line A'B' can be drawn, followed by D'E' with equal opposite slope.

Precisely this technique would give a quick graphical way of tackling the numerical problem just discussed.

First assume a value of M_p. This could be set off from the free bending moments at D and E, so that D'E' could be drawn, and at A, so that A'B' could be drawn. A check would then be made that the largest bending moment elsewhere in the frame (at B_3 in this example) was equal to the assumed value of M_p; If not, a new value of M_p would be guessed and the process repeated.

EXAMPLE: Let use this approach for a Frame pinned base at A and E, with the same loading as previous example

Draw Free diagram at A and E, B.M.=0 Thus A', E' are known



assume $M_p = 54$ and find D' and ~~hence~~ B'

Now check, if needed analytically or graphically and continue the trial-error method to satisfy yield condition i.e. $M \leq M_p$.

For analytical check see the back of the page

Similar to Fixed feet portal we can write

$$A: \quad 96-(M+12S+8T)=0$$
$$B_2: \quad 12-(M+6S) = -M_p$$
$$D: \quad 48-(M-12S) = M_p$$
$$E: \quad 0-(M-12S+8T)=0$$

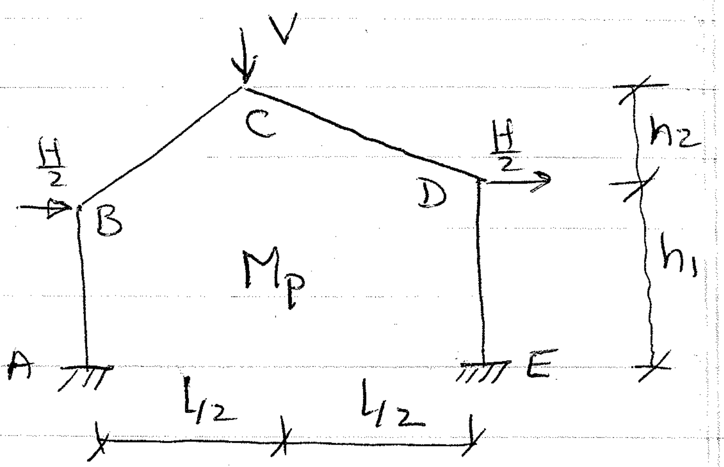
because of pinned feet

$$\therefore \quad M_p = 54$$
$$M = 42$$
$$12S = 48$$
$$8T = 6$$

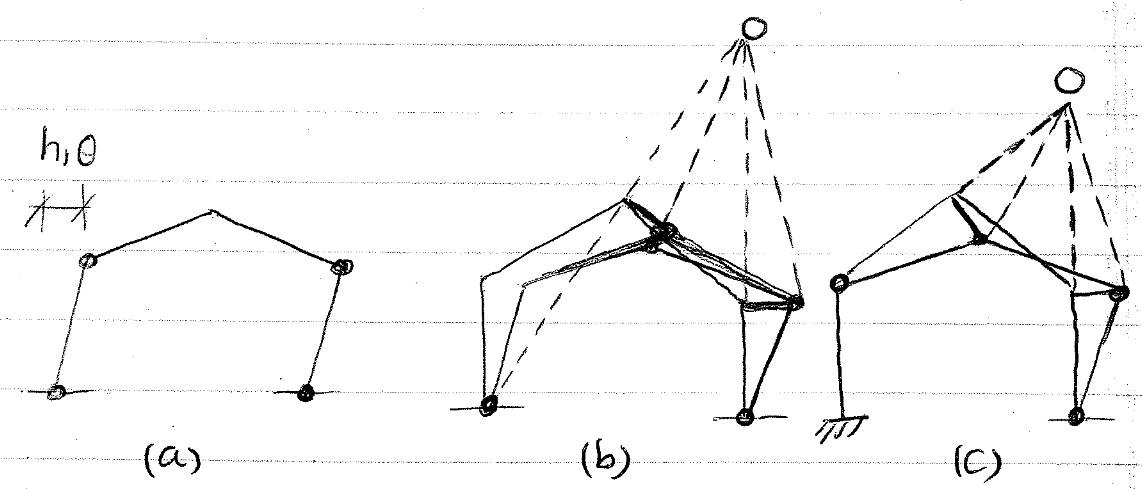
	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0
reactant	−96	−93	−90	−78	−66	−54	−42	−30	−18	−6	6	3	0
total	0	−33	−42	−51	$\boxed{-54}$	−51	−42	−27	−6	21	$\boxed{54}$	39	0

2.9 PITCHED-ROOF PORTAL FRAMES

Calculations are the same as rectangular portal frame. Consider a fixed-base frame as shown with idealised loading. Uniform cross section is considered.



Three basic failure modes are sketched below

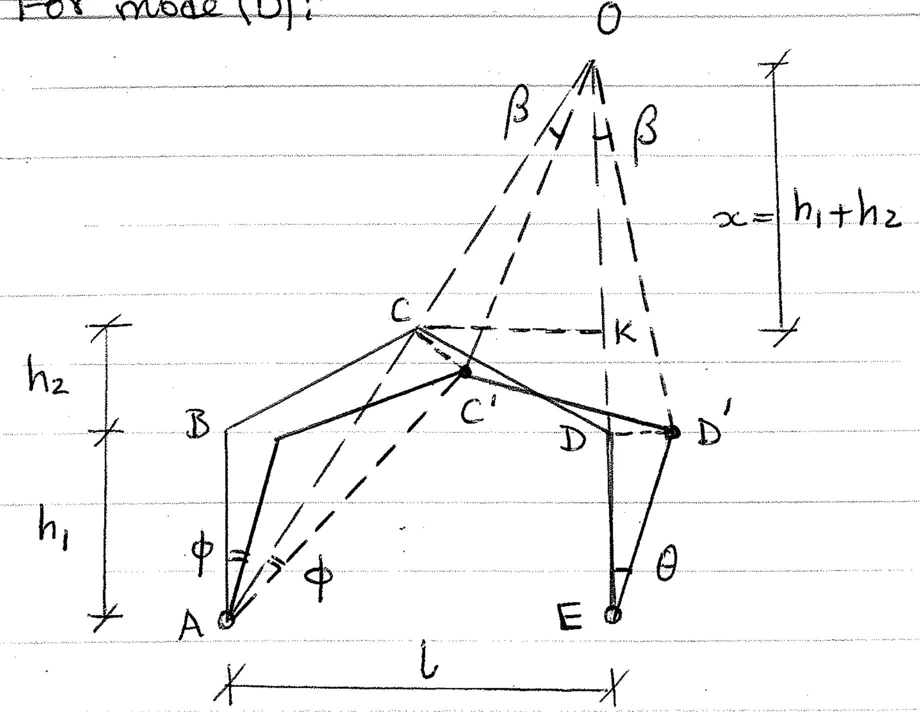


For mode (a) the work equation is easy

$$(h_1\theta)\left(\frac{H}{2}+\frac{H}{2}\right)=(4\theta)(M_p)$$

i.e $$M_p=\frac{1}{4}Hh_1$$

~~For mode (b):~~



$$\frac{CK}{AE} = \frac{x}{x+(h_1+h_2)} = \frac{1}{2}$$

$$x = h_1 + h_2$$

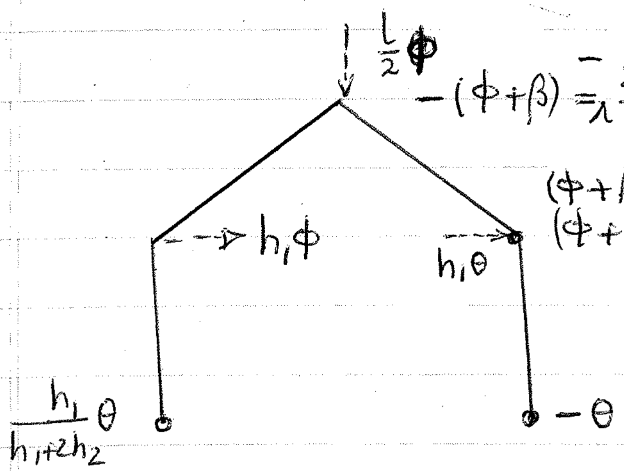
$$\tan\phi = \phi = \frac{CC'}{AC} = \frac{CC'}{OC} = \tan\beta = \beta \qquad \text{i.e} \quad \phi = \beta$$

$$\tan\theta = \frac{DD'}{DE} = \frac{DD'}{h_1} = \theta \qquad \text{but} \quad \tan\beta = \frac{DD'}{OD} = \frac{DD'}{h_1+2h_2} = \beta$$

Equating DD' from the above two relations

$$h_1\theta = (h_1+2h_2)\beta$$

i.e. $\beta = \frac{h_1}{h_1+2h_2}\theta \qquad \phi = \beta = \frac{h_1}{h_1+2h_2}\theta$



$$-(\phi+\beta) = \frac{2h_1}{h_1+2h_2}\theta$$

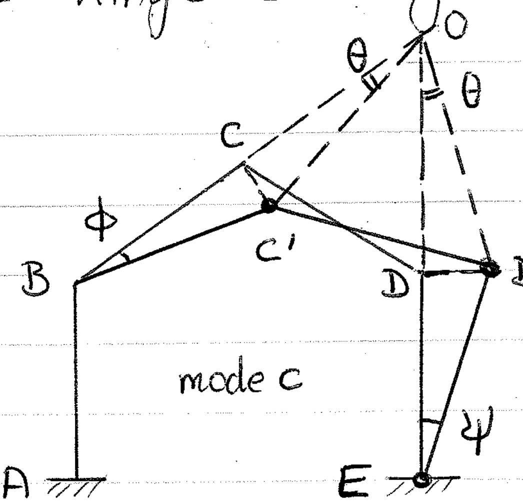
$$(\phi+\beta) =$$

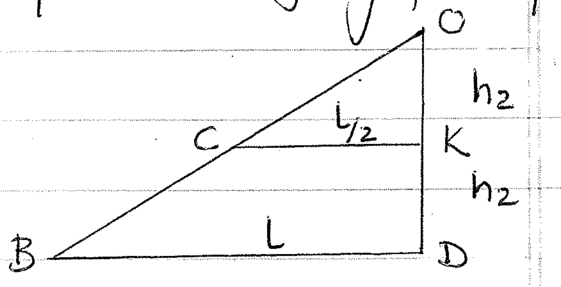
$$(\phi+\theta) = \left(1+\frac{h_1}{h_1+2h_2}\right)\theta = \frac{2(h_1+h_2)}{h_1+2h_2}\theta$$

$$\frac{H}{2}h_1\frac{h_1}{h_1+2h_2}\theta + \frac{H}{2}h_1\theta + \frac{Vl}{2}\frac{h_1}{h_1+2h_2}\theta = M_p\theta\left[\frac{h_1}{h_1+2h_2} + \frac{2h_1}{h_1+2h_2} + \frac{2(h_1+h_2)}{h_1+2h_2} + 1\right] = M_p\theta\left[\frac{6h_1+4h_2}{h_1+2h_2}\right]$$

or $\frac{Vl}{2} + Hh_1\left(1+\frac{h_2}{h_1}\right) = \frac{M_p}{2}\left(6+4\frac{h_2}{h_1}\right)$

Mode (c) is of greatest practical interest, it corresponds to collapse of beam only in the rectangular frame, and it is the most usual mode for real loading (although the hinge C may be displaced slightly from apex).





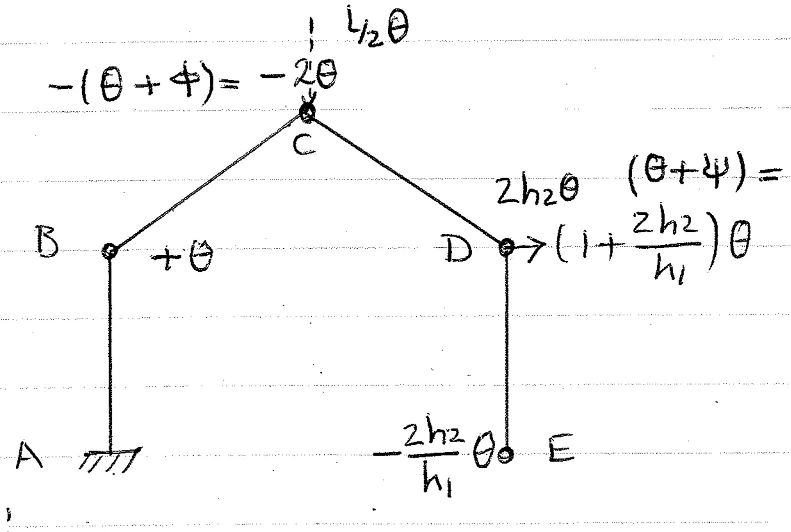
$$\frac{OK}{OD} = \frac{L/2}{L} = \frac{1}{2} = \frac{OK}{OK+KD}$$

$KD = h_2$ hence $OK = h_2$

$$|\psi| = \frac{DD'}{h_1} = \frac{(2h_2)\theta}{h_1} = \frac{2h_2}{h_1}\theta \quad \text{considering sign } \psi = -\frac{2h_2}{h_1}\theta$$

$$\sin\phi = \frac{CC'}{BC} = \frac{CC'}{OC'} = \sin\theta \Rightarrow \phi = \theta$$

Therefore the rotation angles are as follows:



Work Equation:

mode (C)

$$\left(\frac{L}{2}\theta\right)V + (2h_2\theta)\frac{H}{2} = \left(1+2+1+\frac{2h_2}{h_1}+\frac{2h_2}{h_1}\right)\theta M_p$$

$$\frac{VL}{2} + Hh_2 = 4\left(1+\frac{h_2}{h_1}\right)M_p$$

For mode (C):

~~Work equation leads to~~

$$\frac{1}{2}Vl + Hh_2 = \left(4 + 4\frac{h_2}{h_1}\right)M_p$$

For $h_2 = 0$ we get rectangular case. The pitch of the frame ensures that at least one of the columns must participate in the collapse mechanism, and equation above indicates the strengthening effect, namely a factor

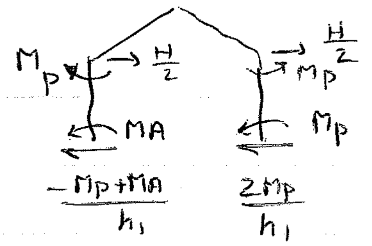
$$\left(1 + \frac{h_2}{h_1}\right)$$

For the collapse mode (C) to be correct, B.M. at A must be less than M_p. The shear balance leads

$$Hh_1 = M_p + M_A$$

$$-M_p \leq Hh_1 - M_p \leq M_p$$

$$0 \leq Hh_1 \leq 2M_p$$



Substituting M_p from above (Top of page) EQ. The condition for collapse by this mode becomes:

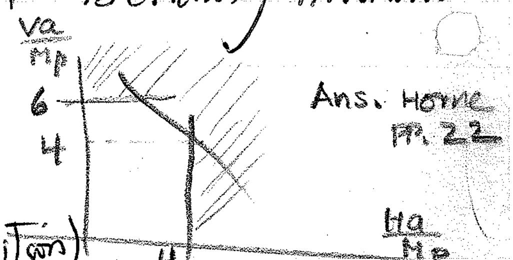
$$0 \leq Hh_1\left(1 + 1/2\,\frac{h_2}{h_1}\right) \leq \frac{1}{4}VL$$

For mode (b) analysis is the same, by considering a rotation θ about the instantaneous centre I_{CD}, which now lies on the intersection of AC and ED. The collapse equation becomes

$$\frac{1}{2}VL + Hh_1\left(1 + 2\frac{h_2}{h_1}\right) = \left(6 + 4\frac{h_2}{h_1}\right)M_p$$

Bending moment diagrams can easily be drawing by summing the free and reactant bending moment diagrams.

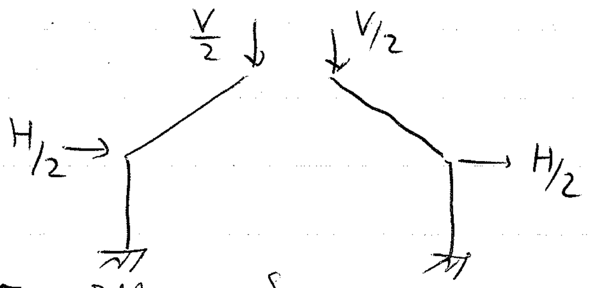
EX: take $l = 2a$, $h_1 = a$ and $h_2 = \frac{a}{2}$ and draw the interaction diagram (Yield surface by our definition)

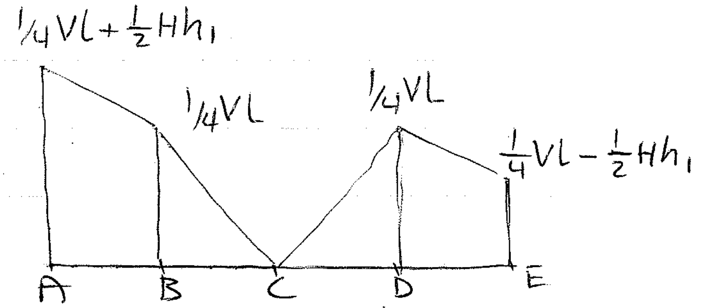


Ans. Horne P. 22

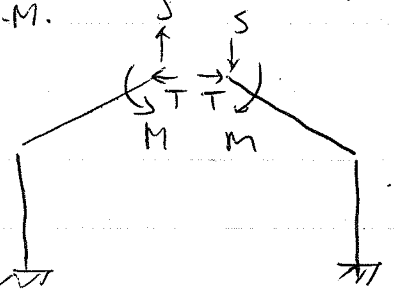
29.1 BENDING MOMENTS

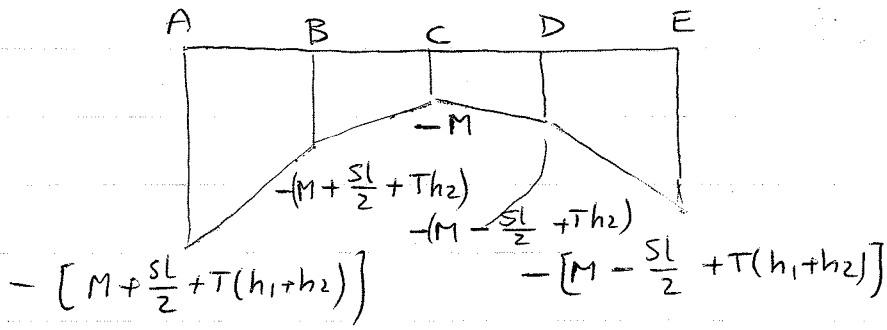
These may be constructed by usual sum of free and reactant frames cut at C.





Free B.M.

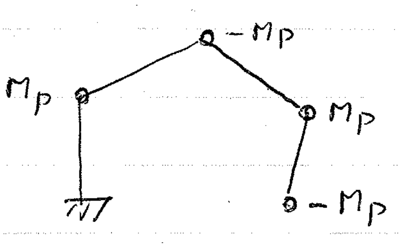




Unknowns; Reactent B.M.

consisting 4 straight lines because of pitch of the roof.

For mode (C)

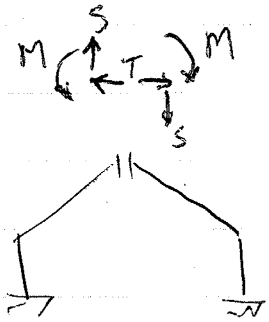


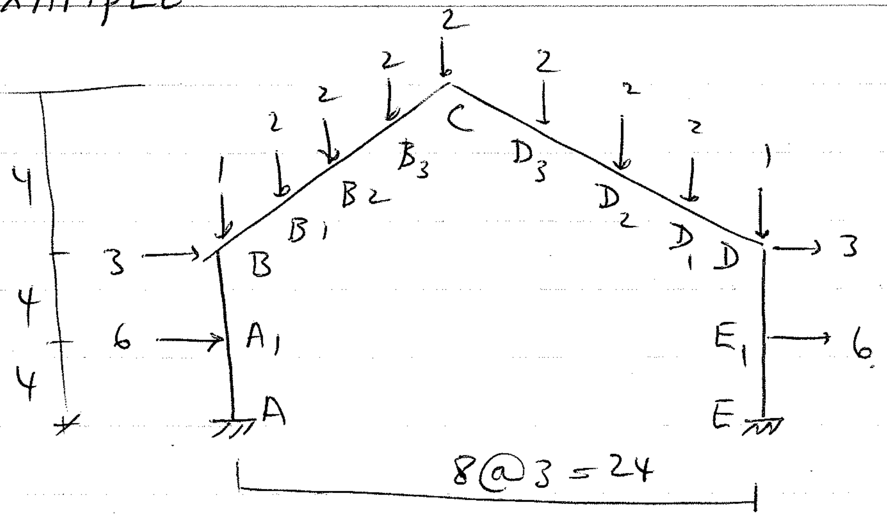
$$B:\ \left(\frac{1}{4}VL\right) - \left(M+\frac{1}{2}SL+Th_2\right) = M_p$$
$$C:\ \qquad - (M) = -M_p$$
$$D:\ \left(\frac{1}{4}VL\right) - \left(M-\frac{1}{2}Sl+Th_2\right) = M_p$$
$$E:\ \left(\frac{1}{4}Vl-\frac{1}{2}Hh_1\right) - \left(M-\frac{1}{2}Sl+Th_1+Th_2\right) = -M_p$$

From solution :

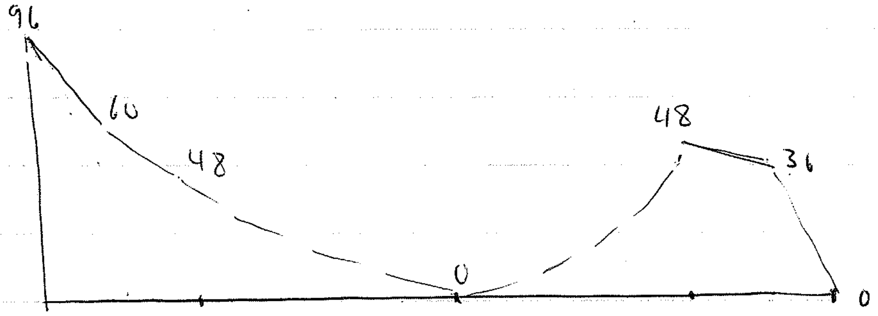
$$\begin{cases} M = M_p \\ S = 0 \\ T(h_1+h_2) = \frac{1}{4}Vl - \frac{1}{2}Hh_1 \end{cases}$$

2.9.2 NUMERICAL EXAMPLE





Free B.M.D.
similar to rect. Frame

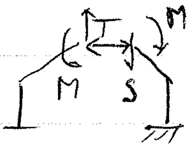


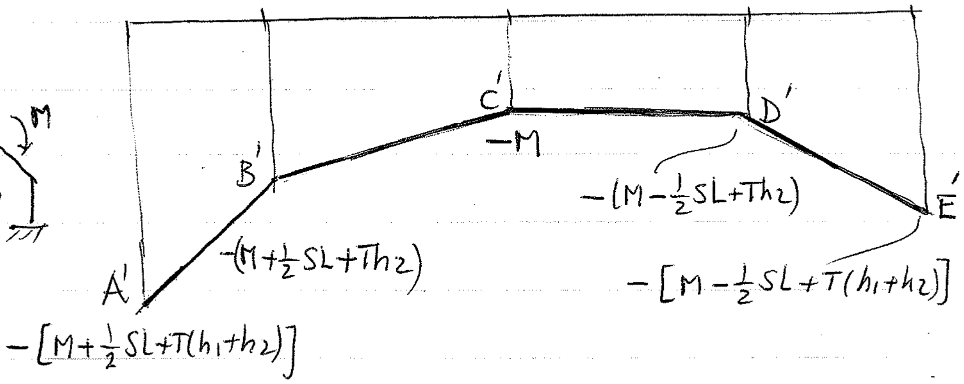
Reactant B.M.s

$\frac{L}{2} = 12$

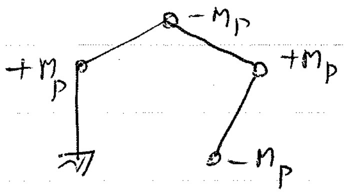
$h_1 = 8$

$h_2 = 4$





For mode (c)



B: $48-(M+12S+4T) = M_p$

C: $0-(M) = -M_p$

D: $48-(M-12S+4T) = M_p$

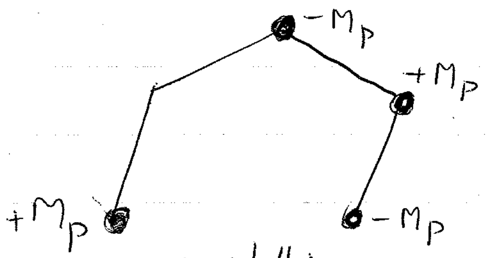
E: $0-(M-12S+12T) = -M_p$

solving these leads to

$$M_p = M = 24$$
$$S = T = 0$$

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D_1	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0
Reactant	−24	−24	−24	−24	−24	−24	−24	−24	−24	−24	−24	−24	−24
Total	$\boxed{72}$	36	24	3	−12	−21	−24	−21	−12	3	24	12	−24

A seriously violated yield condition at A. Thus we try <u>mode (b)</u>



mode (b)

$$A: \quad 96-(M+12S+12T) = M_p$$
$$C: \quad 0-(M) = -M_p$$
$$D: \quad 48-(M-12S+4T) = M_p$$
$$E: \quad 0-(M-12S+12T) = -M_p$$

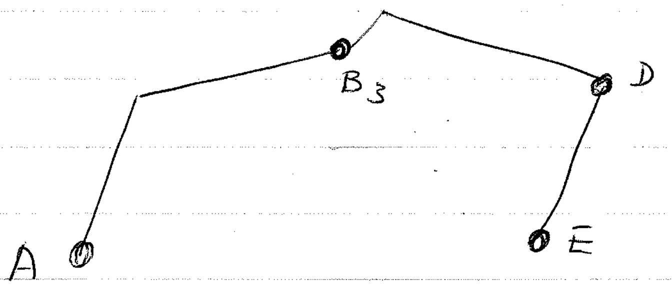
solution gives

$$M_p = M = 30$$
$$12S = 12T = 18$$

Bending moment are now given in Table below;

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	60	0
Reactant	−66	−60	−54	−48	−42	−36	−30	−27	−24	−21	−18	−24	−30
Total	30	0	−6	−21	−30	$\boxed{-33}$	−30	−24	−12	6	30	12	−30

Now solution is almost complete (a small modification) and the correct mechinism is given below



For this mode of collapse we have

$$A: \quad 96-(M+12S+12T)=M_p$$
$$B_3: \quad 3-(M+3S+T)=-M_p$$
$$D: \quad 48-(M-12S+4T)=M_p$$
$$E: \quad 0-(M-12S+12T)=-M_p$$

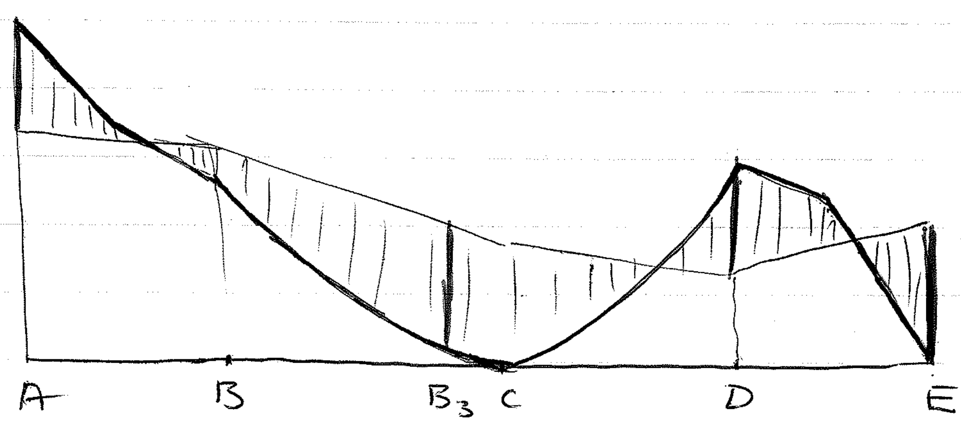
From which

$$M_p = 30.75 \qquad M = 27.75$$
$$12S = 17.25 \qquad 12T = 20.25$$

and the corresponding Table is as

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0
Reactant	-65.25	-58.5	-51.75	-45.75	-39.75	-33.75	-27.75	-25.1	-22.5	-19.9	-17.25	-24	-30.75
Total	30.75	1.5	-3.75	-18.75	-27.75	-30.75	-27.75	-22.1	-10.5	7.1	30.75	12	-30.75

This Table conforms the correctness of the collapse mode, and the final bending moment is as



Final bending moment

The final $M_p = 30.75$ is only marginally less than the corresponding (32.3) for ractangular frame. The strengthening effect of the pitched roof is small for this case, because of relatively high wind load.

2.9.3 PINNED BASE PITCHED ROOF

This case may easily be analyzed by writing down the condition of total be~~ma~~nding moments at both column feet is zero. For this case

$$A: \quad 96-(M+12S+12T)=0$$
$$B_2: \quad 12-(M+6S+2T)=-M_p$$
$$D: \quad 48-(M-12S+4T)=M_p$$
$$E: \quad 0-(M-12S+12T)=0$$

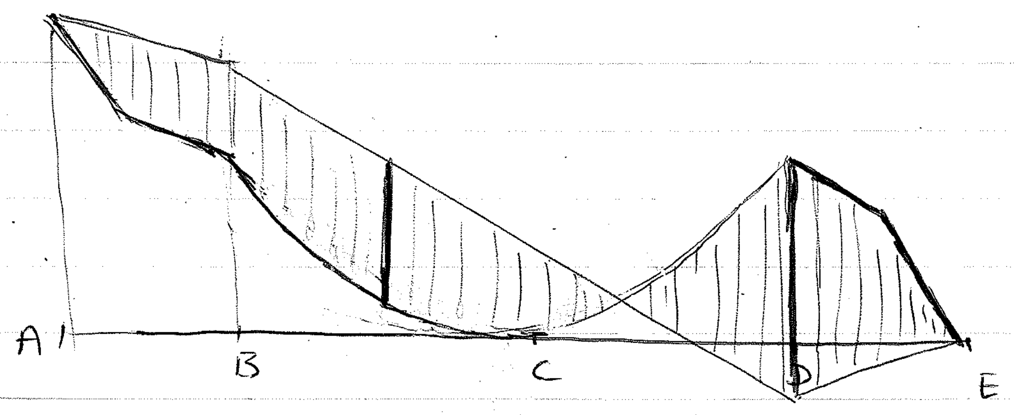
the solution leads to

$$M_p=53.3 \qquad M=40 \qquad 12S=48 \qquad 12T=8$$

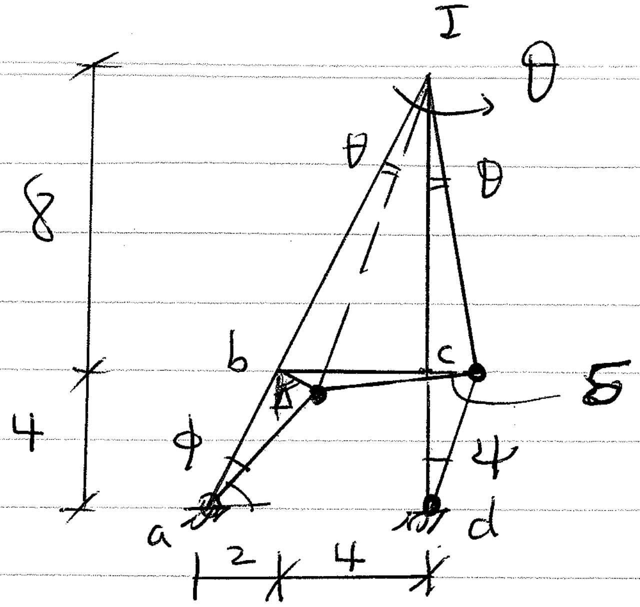
Hence the B.M. Table can be obtained as

	A	A_1	B	B_1	B_2	B_3	C	D_3	D_2	D_1	D	E_1	E
Free	96	60	48	27	12	3	0	3	12	27	48	36	0
reactant	−96	−93.3	−90.7	−78	−65.3	−52.7	−40	−28.7	−17.3	−6	+5.3	2.7	0
total	0	−33.3	−42.7	−51	−53.3	−49.7	−40	−25.7	−5.3	21	53.3	38.7	0

And the corresponding diagram for bending is as



Numerical Example:



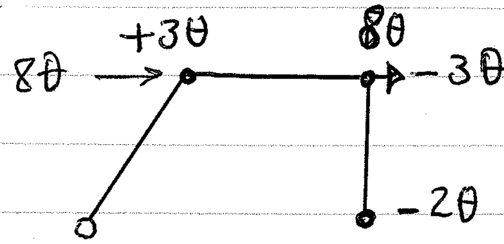
$\psi = \frac{\delta}{4}$ $\theta = \frac{\delta}{8}$ $\boxed{\psi = 2\theta}$ $\boxed{\delta = 8\theta}$

$\phi = \frac{\Delta}{\sqrt{20}}$ $\theta = \frac{\Delta}{2\sqrt{20}}$ $\boxed{\phi = 2\theta}$ $\boxed{\Delta = 2\sqrt{20}\,\theta}$

Rotation of b = 2θ (for ab) + θ (for bc) = 3θ

Rotation of c = θ (for bc) + 2θ (for dc) = 3θ

Thus

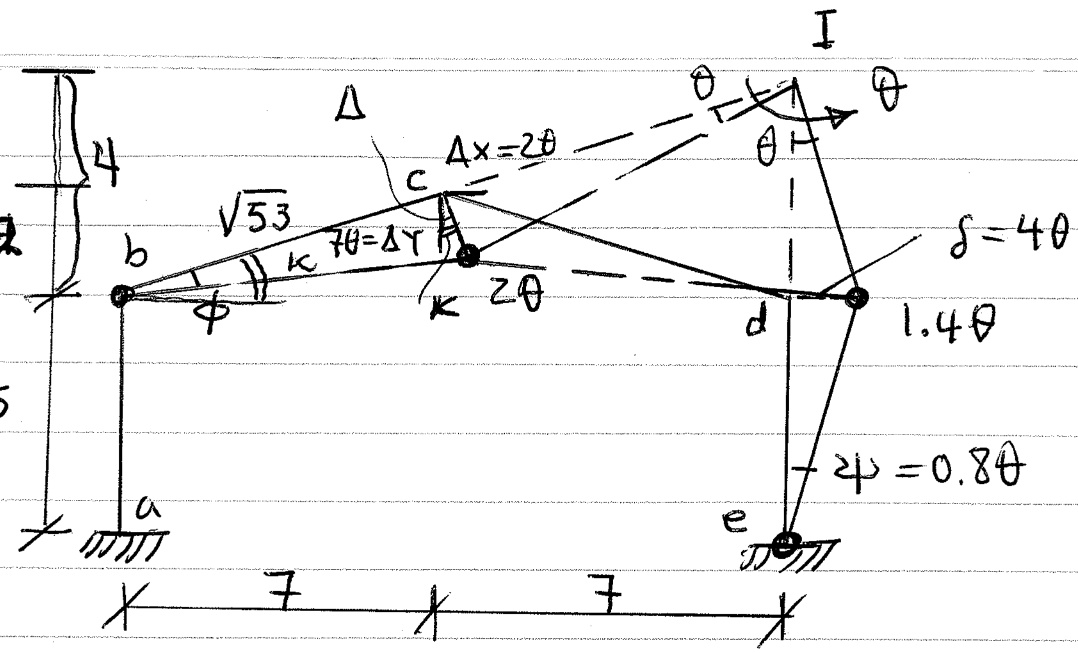


$\Delta = \sqrt{\Delta x^2 + \Delta y^2}$

$\frac{4}{2} = \frac{\Delta x}{\Delta y} = 2$ $\Delta y = \frac{1}{2}\Delta x$ $\Delta = \sqrt{\Delta x^2 + \frac{1}{4}\Delta x^2} = \frac{\sqrt{5}}{2}\Delta x$

$2\sqrt{20}\,\theta = \frac{\sqrt{5}}{2}\Delta x$ $\boxed{\Delta x = 8\theta}$

Example



$\tan\theta = \theta = \frac{\delta}{4}$ $\tan\psi = \frac{\delta}{5} = \psi$

$\frac{\psi}{\theta} = \frac{\frac{\delta}{5}}{\frac{\delta}{4}} = \frac{4}{5} = 0.8$ $\boxed{\psi = 0.8\theta}$ $\boxed{\delta = 4\theta}$

$\phi = \frac{\Delta}{\sqrt{53}}$ $\theta = \frac{\Delta}{\sqrt{53}}$ $\boxed{\phi = \theta}$ $\boxed{\Delta = \sqrt{53}\,\theta}$

$\Delta = \sqrt{\Delta x^2 + \Delta Y^2}$

$\tan k = \frac{2}{7} = \frac{\Delta x}{\Delta Y}$ $\Delta Y = \frac{7}{2}\Delta x$

$\Delta = \sqrt{\Delta x^2 + \frac{49}{4}\Delta x^2} = \frac{\sqrt{53}}{2}\Delta x$

$\sqrt{53}\,\theta = \frac{\sqrt{53}}{2}\Delta x$ $\boxed{\Delta x = 2\theta}$ $\boxed{\Delta Y = 7\theta}$

Rotation of cd is θ (This can also be proved)

change of angle at c = θ (for bc) + θ (for cd) = 2θ

change of angle at d = 0.8θ (for de) + θ (for dc)

= 1.4θ