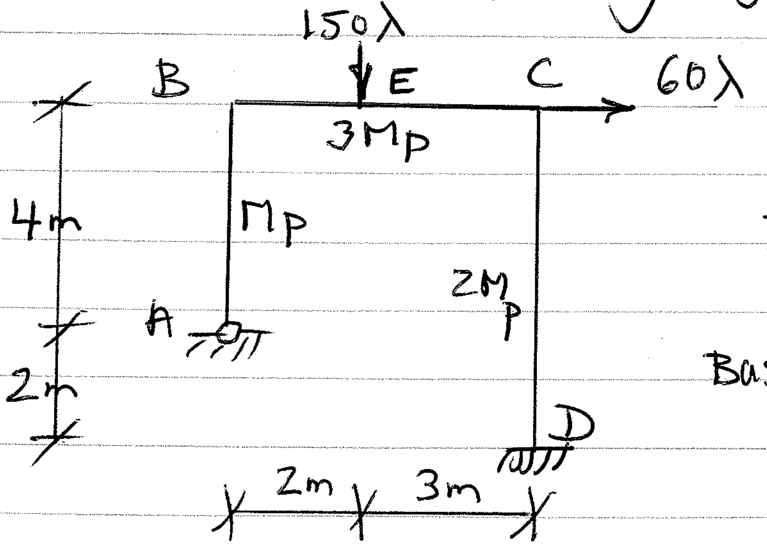
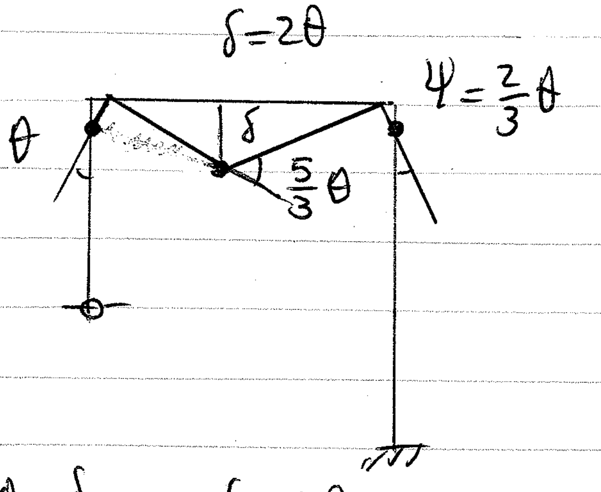
Example 6: Consider the frame ABCD. Given $M_p = 120$ kN.m and the loading system, find λ_c.

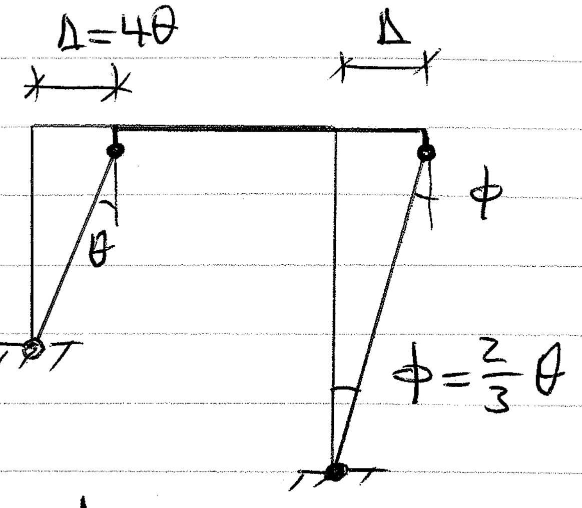


$\gamma(S) = 2$

$N = 4$

Basic Mechs $= 4 - 2 = 2$





$\theta = \frac{\delta}{2} \qquad \delta = 2\theta$

$\psi = \frac{\delta}{3} = \frac{2}{3}\theta$

$\theta = \frac{\Delta}{4} \qquad \Delta = 4\theta$

$\phi = \frac{\Delta}{6} = \frac{4\theta}{6} = \frac{2\theta}{3}$

$M_p(\theta) + 3M_p(\theta + \frac{2}{3}\theta) + 2M_p(\frac{2}{3}\theta) = 150\lambda_1(2\theta)$

$M_p[\theta + \frac{15}{3}\theta + \frac{4}{3}\theta] = 300\lambda_1\theta$

$\frac{22}{3}M_p = 300\lambda_1 \qquad \lambda_1 = \frac{22 \times 120}{900} = \underline{2.933}$

$M_p(\theta) + 2M_p(\frac{2}{3}\theta) + 2M_p(\frac{2}{3}\theta) = 60\lambda(4\theta)$

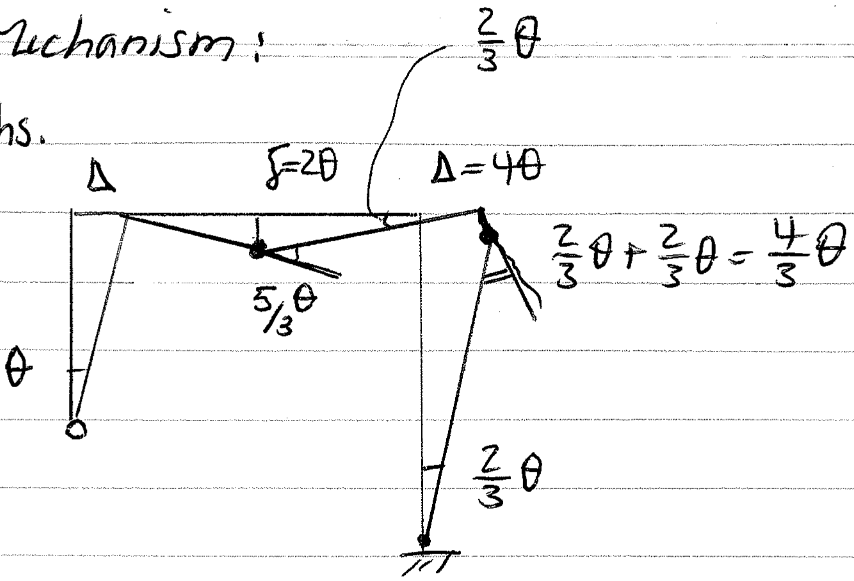
$M_p[\theta + \frac{4}{3}\theta + \frac{4}{3}\theta] = 240\lambda_2\theta$

$\frac{11}{3}M_p = 240\lambda_2$

$\lambda_2 = \frac{11 \times 120}{240 \times 3} = \underline{1.833}$

Combined Mechanism:

Add TWO mechs. to obtain



$3M_p\left(\frac{5}{3}\theta\right) + \left(\frac{4}{3}\theta\right)(2M_p) + \left(\frac{2}{3}\theta\right)(2M_p)$

$= (150\lambda)(2\theta) + (60\lambda)(4\theta)$

$M_p\left[5\theta + \frac{8}{3}\theta + \frac{4}{3}\theta\right] = \lambda\theta\left[300 + 240\right]$

$9M_p = \lambda(540)$ $\qquad \lambda_3 = \frac{9 \times 120}{540} = \underline{2.00}$

Therefore $\boxed{\lambda_c = 1.833}$

Yield check: For the second mod we check the yield at E. We use the first mod as a help mechanism and write virtual work.

$\theta \times M_p + \frac{5}{3}\theta \times M_E + \frac{2}{3}\theta \times 2M_p$

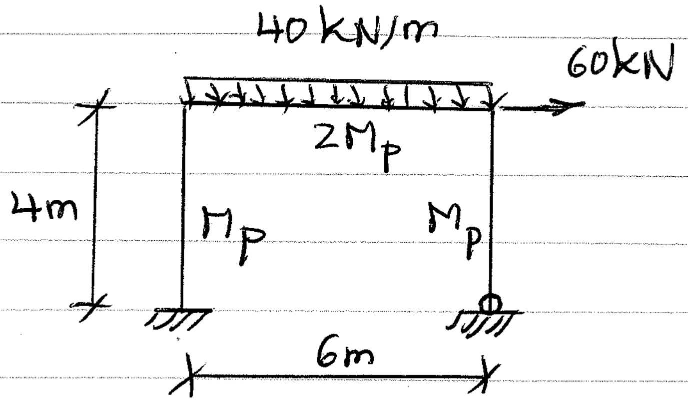
$= 300\lambda\theta$

$\frac{7}{3}M_p\theta + \frac{5}{3}M_E\theta = 550\theta$

$M_E = \frac{3}{5} \times 270 = 162 \qquad < 3M_p = 360$

Yield is satisfied.

Example: For the frame shown find λ_c

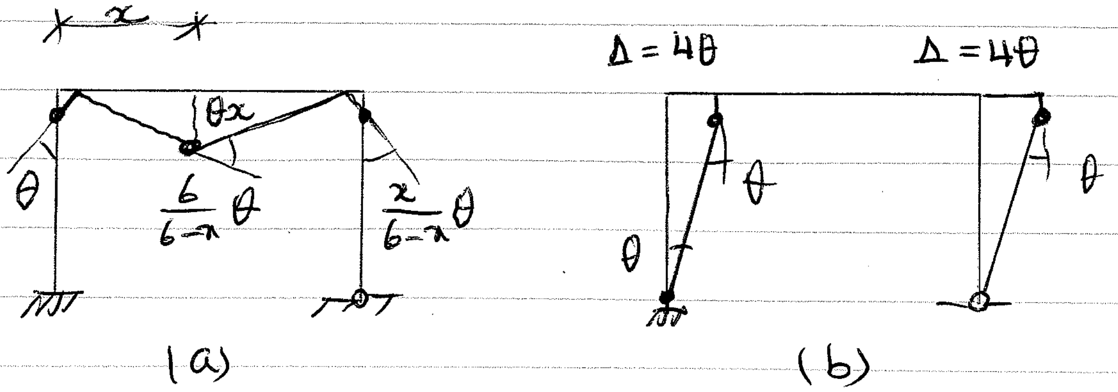


$M_p = 160$ is given

$\gamma(S) = 2$

$N = 4$

$4 - 2 = 2$ basic mechs.



For Mechanism (a) we have

$$(\theta)(160) + \left(\frac{6}{6-x}\theta\right)(320) + \left(\frac{x}{6-x}\theta\right)(160)$$

$$= \frac{1}{2} \times x\theta \times 6 \times 40\lambda$$

$$160\left(\frac{18}{6-x}\right)\theta = 120\lambda x\theta \qquad \lambda_1 = \frac{4}{3}\,\frac{18}{(6-x)x}$$

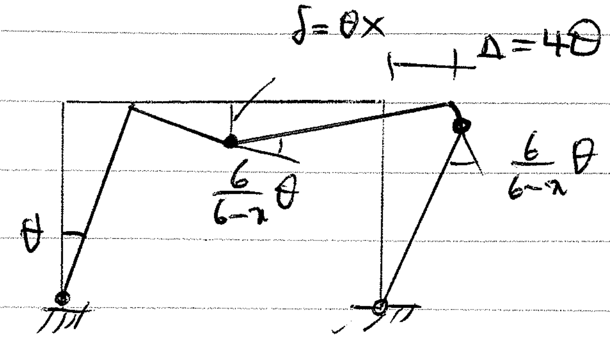
for minimum value of λ_1 $\quad \frac{d\lambda_1}{dx} = 0 \Rightarrow x = 3$

Hence $\quad \underline{\lambda_{min} = 2.67}$

For Mechanism (b) we have

$$3(\theta)(160) = 40\lambda\, 60 \qquad \lambda_2 = \frac{480}{4 \times 60} = \underline{2.0}$$

Combining (a) and (b) we have (c) as shown



(c)

$$(\theta)(160) + \left(\frac{6}{6-x}\theta\right)320 + \left(\frac{6}{6-x}\theta\right)(160) = 120\lambda(x+2)\theta$$

$$160\left(\frac{24-x}{6-x}\right)\theta = 120\lambda(x+2)\theta$$

$$\lambda_3 = \frac{4}{3}\left[\frac{(24-x)}{(6-x)(x+2)}\right] \qquad \frac{d\lambda_3}{dx} = 0 \Rightarrow x = 2.367$$

$$\underline{\lambda_{3\,min} = 1.818}$$

This is the least λ and may well be λ_c.

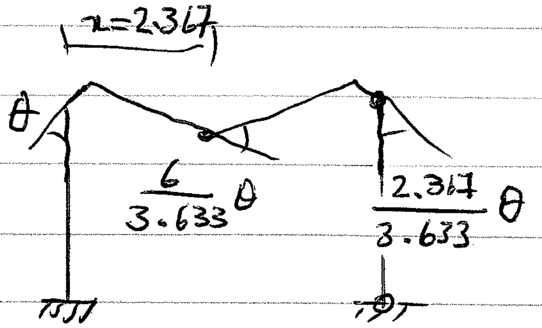
Now Yield is checked at B.
Using the result of mod (a) as a help mech.
We have

$$-(\theta)(M_B) + \left(\frac{6}{3.633}\theta\right)(320) + \left(\frac{2.367}{3.633}\theta\right)(160)$$

$$= \frac{1}{2} \times 2.367\theta \times 6 \times 1.818 \times 40$$

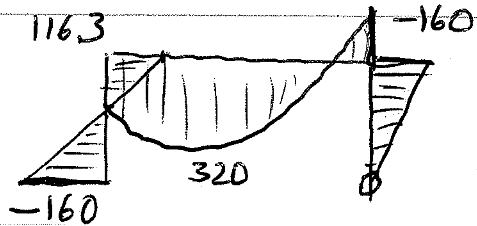
$$(632.73 - M_B)\theta = 516.38\theta$$

$M_B = 116.34$ kN.m $< M_p = 160$

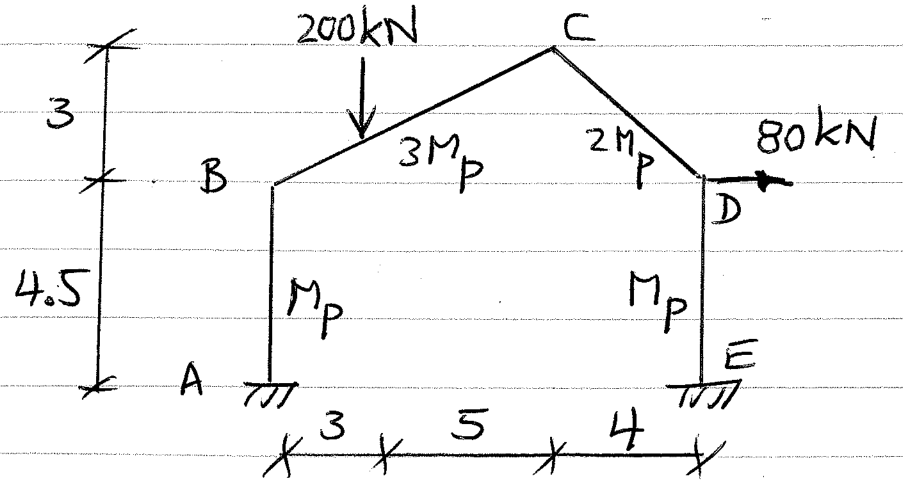


(a)

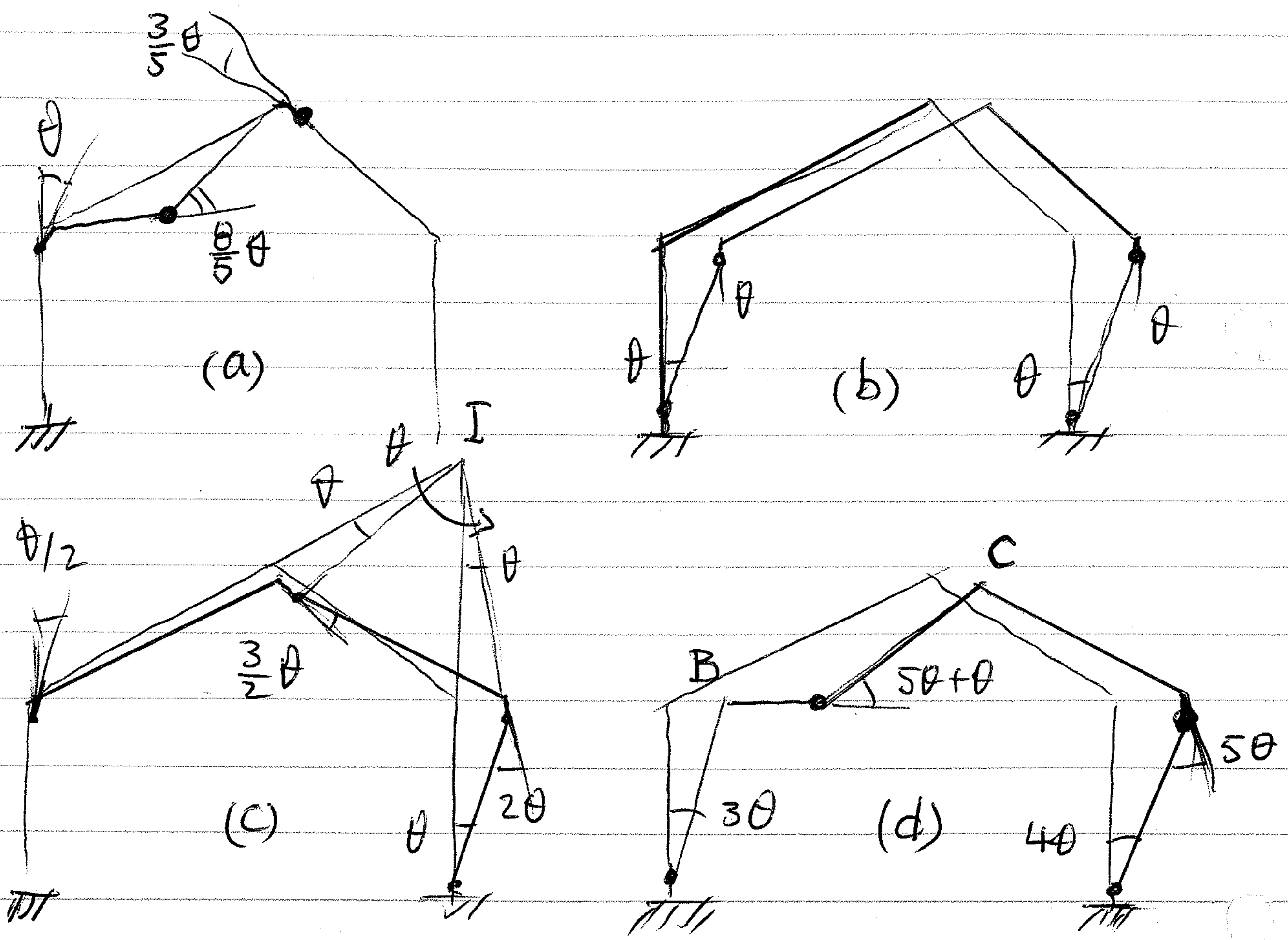
Bending moment is as shown



Example: Find M_p such that under the Loading shown the frame just collapses



$\gamma(s) = 3$ $\quad N = 6 \quad$ $N - \gamma(s) = 6 - 3 = 3$ basic Mechs



For mech. (a)

$$\theta \times M_p + \frac{8}{5}\theta \times 3M_p + \frac{3}{5}\theta \times 2M_p = 3\theta \times 200$$

$$M_p = \frac{600}{7} = \underline{85.7} \text{ kN.m}$$

For mech. (b)

$$(\theta+\theta+\theta+\theta)\,M_p = 4.5\theta \times 80$$

$$4M_p\theta = 360\theta$$

$$M_p = \frac{360}{4} = \underline{90.0} \text{ kN.m}$$

For mech. (c)

$$\left(\frac{\theta}{2}\right)M_p + \left(\frac{\theta}{2}+\theta\right)2M_p + (2\theta+\theta)\,M_p =$$

$$3 \times \frac{\theta}{2} \times 200 + 4.5\theta \times 80$$

$$M_p = \frac{660}{6.5} = \underline{101.5} \text{ kN.m}$$

For combined mechanism (d) we have

$$(3\theta + 4\theta + 4\theta + \theta)\,M_p + (4\theta)\,3M_p =$$

$$\frac{5}{2} \times 600\,\theta + 3 \times 360\theta + 660\,\theta$$

$$24 M_p\theta = 3240\,\theta \qquad M_p = \frac{3240}{24} = \underline{135} \text{ kN.m}$$

We check the bending moments at B & C for the combined mechanism (d).

Using The mod (b) as a help mehanism we have:

$$(\theta)(135)+(\theta)(M_B)+(\theta+\theta)135 = 360\,\theta$$
$$(M_B+405)\theta = 360\,\theta$$
$$M_B = 360-405 = -45 < 135$$

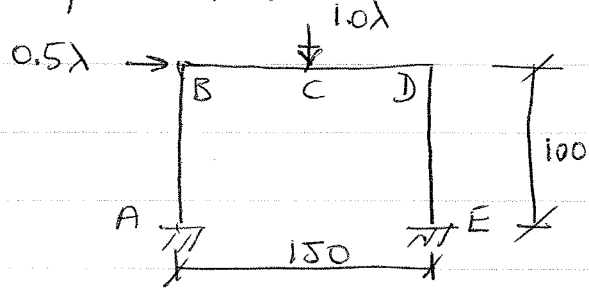
Using the mod (a) as a help mechanism we have:

$$(\theta)(45)+(\tfrac{8}{5}\theta)\times3\times135+(\tfrac{3}{5}\theta M_c) =$$
$$(693+0.6M_c)\,\theta$$
$$M_c = \frac{600-693}{0.6} = -155 < 270$$

Thus Yield is also satisfied and $M_p = 135$ kN.m is The answer.

4.5 METHOD OF INEQUALITIES

This method is due to Neal and Symonds and is based on a technique proposed by Dines. This will be illustrated with reference to a portal frame of uniform cross section with M_p as 40 units. We want to find λ.



Yield condition will be satisfied if

$$-40 \leq M_A, M_B, M_C, M_D, M_E \leq 40$$

Thus there are 10 inequalities which can be expressed as

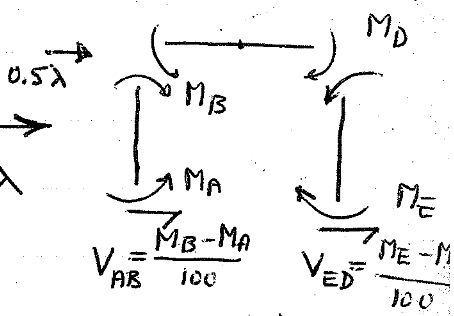
(1)	$M_A + 40 \geq 0$	$-M_A + 40 \geq 0$	(2)
(3)	$M_B + 40 \geq 0$	$-M_B + 40 \geq 0$	(4)
(5)	$\underline{M_C} + 40 \geq 0$	$\underline{-M_C} + 40 \geq 0$	* (6)
(7)	$M_D + 40 \geq 0$	$-M_D + 40 \geq 0$	(8)
(9)	$\underline{M_E} + 40 \geq 0$	$\underline{-M_E} + 40 \geq 0$	(10)

Equilibrium equations can be written as

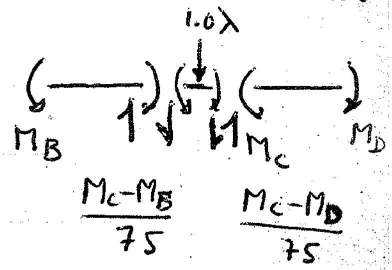
(11) $\sum F_y = 0 \Rightarrow \quad M_C - \frac{M_B}{2} - \frac{M_D}{2} = \frac{150\lambda}{4}$

and

(12) $\sum F_x = 0 \Rightarrow \quad -M_A + M_B - M_D + M_E = 50\lambda$



If two B.Ms, say $\underline{M_C}$ and $\underline{M_E}$ are found from (11) & (12) and put in (5),(6),(9),(10), inequalities become as follows:



$$M_B + M_D + 80 + 75\lambda \geq 0 \qquad (13)$$
$$-M_B - M_D + 80 - 75\lambda \geq 0 \qquad (14)$$
$$M_A - M_B + M_D + 40 + 50\lambda \geq 0 \qquad (15)$$
$$-M_A + M_B - M_D + 40 - 50\lambda \geq 0 \qquad (16)$$

Any combination of M_A, M_B and M_D will satisfy the conditions of equilibrium, provided M_C and M_E are calculated from (11) and (12). Hence the collapse load factor is the greatest value of λ for which the following 10 inequalities can be satisfied:

Maximize λ

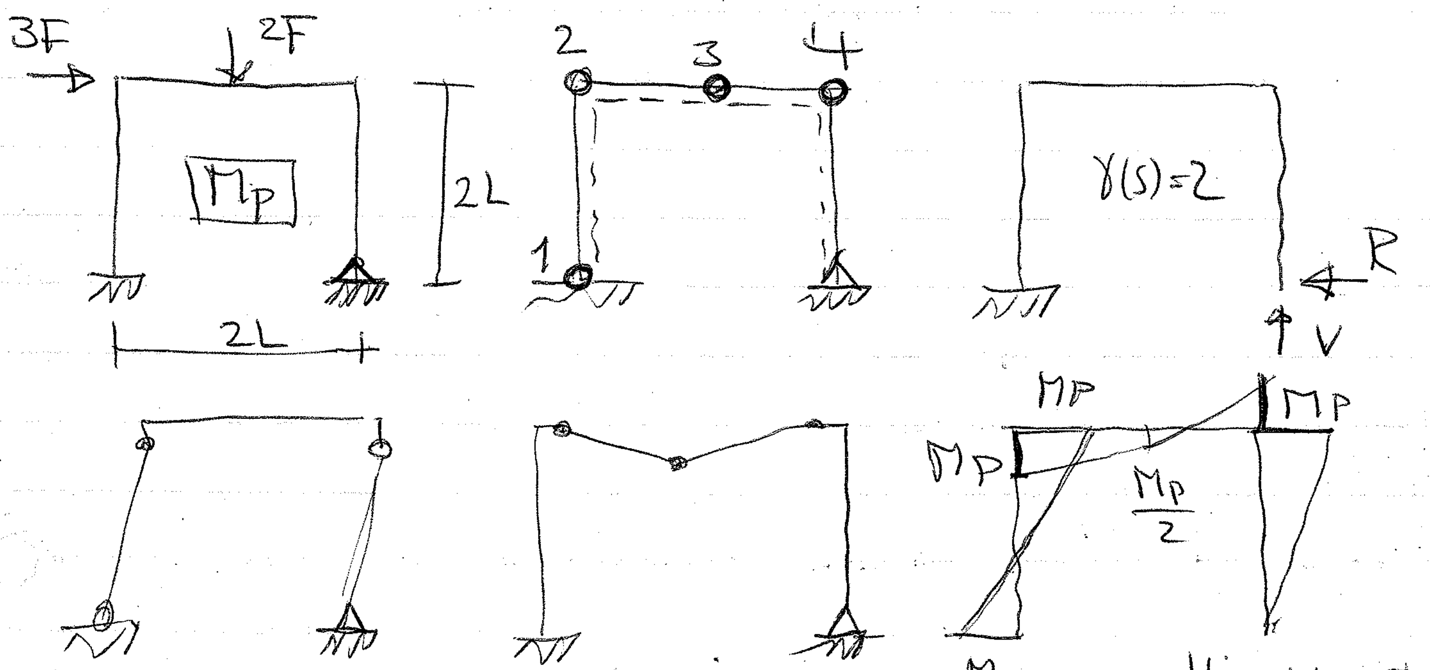
Subject to

$$M_A + 40 \geq 0 \qquad -M_A + 40 \geq 0$$
$$M_B + 40 \geq 0 \qquad -M_B + 40 \geq 0$$
$$M_B + M_D + 80 + 75\lambda \geq 0 \qquad -M_B - M_D + 80 - 75\lambda \geq 0$$
$$M_D + 40 \geq 0 \qquad -M_D + 40 \geq 0$$
$$M_A - M_B + M_D + 40 + 50\lambda \geq 0 \qquad -M_A + M_B - M_D + 40 - 50\lambda \geq 0$$

The critical value of λ may be obtained by eliminating the three bending moments M_A, M_B and M_D in turn [see sld skeleton]. Later more proper method will be presented

EXAMPLE

Consider the following frame. Four locations for possible plastic hinges are shown. $\gamma(s) = 2$ and it is possible to express the moments at each potential plastic hinge locations as a function of two arbitrarily selected redundant forces. Selecting V and R as redundants and using the sign convention shown, the moments at hinge locations can be written as



Basic Mechanisms

resulting Moment diagram

$$M_4 = -2RL$$
$$M_3 = -2RL + VL$$
$$M_2 = -2RL + 2VL - 2FL$$
$$M_1 = 2VL - 2FL - 6FL$$

We have 4 equations with 6 unknowns.

Using manipulations two redundant forces can be eliminated, to obtain

$$-M_2 + 2M_3 - M_4 = 2FL$$
$$-M_1 + M_2 - M_4 = 6FL$$

These two equations are equilibrium equations, for the structure.

We could obtain those two equations by virtual work for basic mechanisms.

$$-M_2\theta + 2M_3\theta - M_4\theta = 2FL\theta$$
$$M_2\theta - M_1\theta - M_4\theta = 6FL\theta$$

However, for theses basic ~~stech~~ mechanisms, the hinge moment M_i are not expressed in terms of M_p, but rather unknown moments whose values should be determined.

Now we still have 2 equations with 4 unknowns. In plastic design we have additional constraints as follows:

$$-M_p \leq M_1 \leq M_p$$
$$-M_p \leq M_2 \leq M_p$$
$$-M_p \leq M_3 \leq M_p$$
$$-M_p \leq M_4 \leq M_p$$

Now we have to search for the largest possible applied loads while respecting the above equilibrium equations and inequalities.

One can perform the following algorithm:

Use equilibrium equations and the inequality conditions to systematically eliminate the unknowns, one by one, performing all possible cross-comparisons of inequality equations, until all remaining equations are expressed in terms of known quantities. One can then find the largest possible value for the applied loads (often expressed as a function of a common load F) simply by scanning all resulting inequalities. The largest value of F that satisfies all resulting constraints gives the collapse load.

For the example at hand, find two unknown in terms of the others as

$$M_3 = FL + \frac{1}{2}(M_2 + M_4)$$

$$M_1 = -6FL + M_2 - M_4$$

Substituting in inequalities we find:

$$-M_p \leq -6FL + M_2 - M_4 \leq M_p$$

$$-M_p \leq M_2 \leq M_p$$

$$-M_p \leq FL + \frac{1}{2}(M_2 + M_4) \leq M_p$$

$$-M_p \leq M_4 \leq M_p$$

Now further elimination of unknowns through the inequatity relationships proceeds by Construction of all possible inferences from the inequality set. For example, to eliminate M_2 from the inequality set, all inequalities containing M_2 must be rearranged to isolate that term within the inequalities. This gives

$$-M_P \leq M_2 \leq M_P \qquad (a)$$

$$-2M_P-2FL-M_4 \leq M_2 \leq 2M_P-2FL-M_4 \qquad (b)$$

$$-M_P+6FL+M_4 \leq M_2 \leq M_P+6FL+M_4 \qquad (c)$$

Then the left side of the inequalities are compared with the right side of the same inequalities, i.e the left side of (a) is compared to right-side of (a), (b) and (c). The left of (b) is compared with the same three right side of inequalities. Similarly the left of (c) compared to right side of (a), (b) and (c).

$$-M_P \leq M_P \qquad (a)$$

$$-M_P \leq 2M_P-2FL-M_4 \qquad (b)$$

$$-M_P \leq M_P+6FL+M_4 \qquad (c)$$

$$-2M_P-2FL-M_4 \leq M_P \qquad (d)$$

$$-2M_P-2FL-M_4 \leq 2M_P-2FL-M_4 \qquad (e)$$

$$-2M_P-2FL-M_4 \leq M_P+6FL+M_4 \qquad (f)$$

$$-M_P+6FL+M_4 \leq M_P \qquad (g)$$

$$-M_P+6FL+M_4 \leq 2M_P-2FL-M_4 \qquad (h)$$

$$-M_P+6FL+M_4 \leq M_P+6FL+M_4 \qquad (i)$$

a, e and i are automatically satisfied. By finding matching pairs we can write:

(c) and (g) $\Rightarrow$ $$-2M_p - 6FL \leq M_4 \leq 2M_p - 6FL \qquad (a)$$
(b) and (d) $\Rightarrow$ $$-3M_p - 2FL \leq M_4 \leq 3M_p - 2FL \qquad (b)$$
(f) and (h) $\Rightarrow$ $$-\tfrac{3}{2}M_p - 4FL \leq M_4 \leq \tfrac{3}{2}M_p - 4FL \qquad (c)$$
$$-M_p \leq M_4 \leq M_p \qquad (d)$$

NOW we repeat the process to eliminate M_4.

$$-2M_p - 6FL \leq 2M_p - 6FL \qquad (a)$$
$$-2M_p - 6FL \leq 3M_p - 2FL \qquad (b)$$
$$-2M_p - 6FL \leq \tfrac{3}{2}M_p - 4FL \qquad (c)$$
$$-2M_p - 6FL \leq M_p \qquad (d)$$
$$-3M_p - 2FL \leq 2M_p - 6FL \qquad (e)$$
$$-3M_p - 2FL \leq 3M_p - 2FL \qquad (f)$$
$$-3M_p - 2FL \leq \tfrac{3}{2}M_p - 4FL \qquad (g)$$
$$-3M_p - 2FL \leq M_p \qquad (h)$$
$$-\tfrac{3}{2}M_p - 4FL \leq 2M_p - 6FL \qquad (i)$$
$$-\tfrac{3}{2}M_p - 4FL \leq 3M_p - 2FL \qquad (j)$$
$$-\tfrac{3}{2}M_p - 4FL \leq \tfrac{3}{2}M_p - 4FL \qquad (k)$$
$$-\tfrac{3}{2}M_p - 4FL \leq M_p \qquad (l)$$
$$-M_p \leq 2M_p - 6FL \qquad (m)$$
$$-M_p \leq 3M_p - 2FL \qquad (n)$$
$$-M_p \leq \tfrac{3}{2}M_p - 4FL \qquad (o)$$
$$-M_p \leq M_p \qquad (p)$$

Eliminating a, f, k and p provides no useful information.

The following 12 inequalities are obtained for F, respectively:

$$-\frac{5M_p}{4L} \leq F$$

$$-\frac{7M_p}{4L} \leq F$$

$$-\frac{M_p}{2L} \leq F$$

$$F \leq \frac{5M_p}{4L}$$

$$F \leq \frac{9M_p}{4L}$$

$$-2M_p \leq F$$

$$F \leq \frac{7M_p}{4L}$$

$$-\frac{9M_p}{4L} \leq F$$

$$-\frac{5M_p}{8L} \leq F$$

$$F \leq \frac{M_p}{2}$$

$$F \leq \frac{2M_p}{L}$$

$$F \leq \frac{5M_p}{8L}$$

The range of F satisfying all the above inequalities is

$$-\frac{M_p}{2L} \leq F \leq \frac{M_p}{2L}$$

or $F = \left| \frac{M_p}{2L} \right|$.

Substitution in the previous inequalities results in the values for the unknowns.

$$(-2M_p - 3M_p = -5M_p) \leq M_4 \leq (-M_p = 2M_p - 3M_p) \quad (a)$$
$$-4M_p \leq M_4 \leq 2M_p \quad (b)$$
$$-3.5M_p \leq M_4 \leq -0.5M_p \quad (c)$$
$$-M_p \leq M_4 \leq M_p \quad (d)$$

(a) ad (d) results in $M_4 = -M_p$ indicating the presence of a plastic hinge at that location.

Substituting this result and the F obtained leads to

$$-M_p \leq M_2 \leq M_p$$
$$(-2M_p - 2(\tfrac{1}{2})M_p - (-M_p) = -2M_p) \leq M_2 \leq (2M_p = 2M_p - 2(\tfrac{1}{2})M_p - (-M_p))$$
$$M_p \leq M_2 \leq 3M_p$$

Therefore $M_2 = M_p$.

Now substituting in equilibrium equations

$$M_3 = \tfrac{1}{2}M_p + \tfrac{1}{2}(M_p - M_p) = \tfrac{1}{2}M_p$$
$$M_1 = -6(\tfrac{1}{2})M_p + M_p - (-M_p) = -M_p$$

Now the final diagram for bending is obtained

For computerized methods see:

1. R.K. Livesley, Matrix Methods of Structural Analysis, Pergamon Press, UK, 1975.

General case

N: number of critical section

$\gamma(s)$: DSI

Number of equilibrium equations $= N - \gamma(s) = D$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & \cdots & & \\ & & & \\ c_{D1} & \cdots & & c_{DN} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_N \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix} P$$

A series of N inequalities as

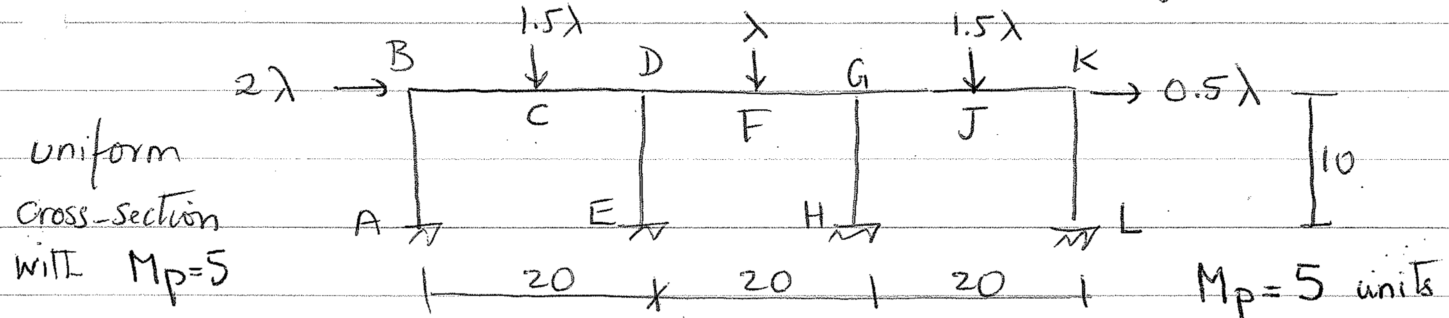
$$-M_{P1} \le M_1 \le M_{P1}$$
$$-M_{P2} \le M_2 \le M_{P2}$$
$$\vdots$$
$$-M_{PN} \le M_N \le M_{PN}$$

A Linear program maximizing P will result in the required solution.

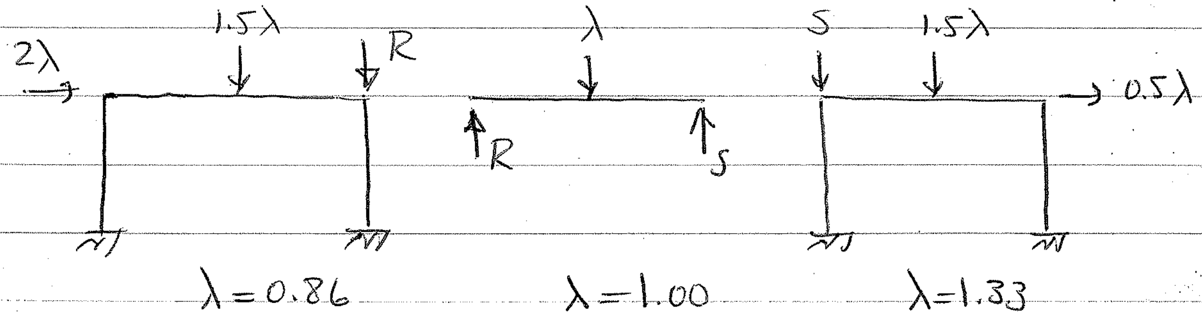
4.6 ANALYSIS BY ADJUSTMENT OF CONSTRAINTS

As we know, if either an internal or external constraint is removed from the structure, the collapse load factor remains constant or decreases. Heyman and Nachbar suggested the use of this fact to derive safe estimates of λ since the removal of restraints usually facilitates analysis.

The method is illustrate by the following example.

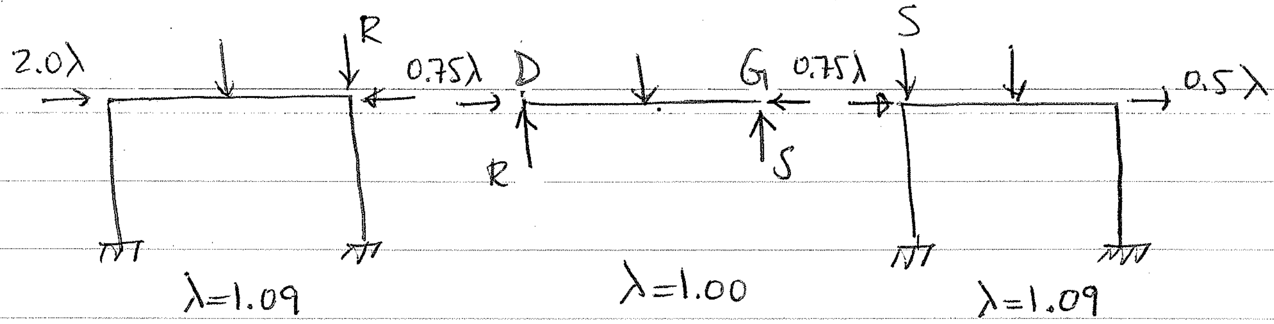


The structures is decomposed as



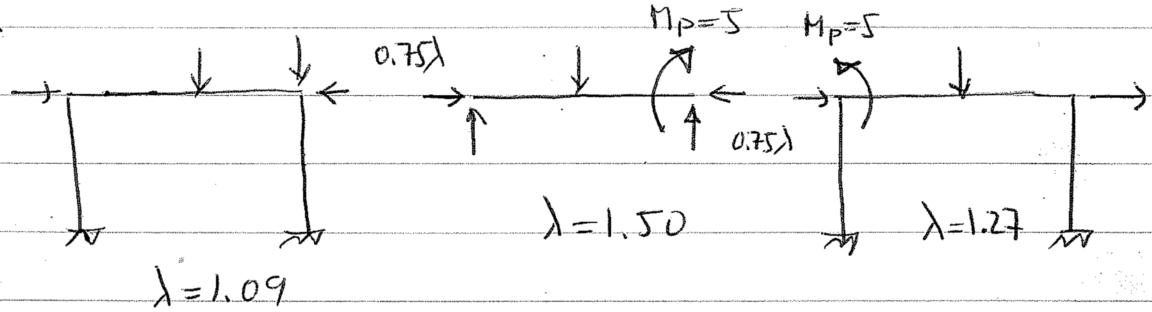
We calculate λ independently. If structure behaves in this way its collapse load would be the smallest of the above values; i.e $\lambda = 0.86$ which represents a safe estimate of the collapse load.

If the horizontal load of 2.5λ is equally divided and considered as axial load for the beams, then we have



2.5λ is equally divided between two portals i.e 1.25λ for each
The smallest load factor is λ=1.00

The next step is to improve the load factor for the beam by introducing hogging bending moments at one or both ends of DG. Inspection shows that a hogging bending moment at D would decrease the load factor for left hand portal frame, whereas it will increase for right hand portal. Thus a hogging moment equal to 5 units (M_p) is introduced at G.



The correct load factor at collapse may be obtained by continuing the process until it is impossible to increase the minimum load factor by adjusting the reaction and bending moments between the components of the structure.

In this example, the true load factor is $\lambda_c=1.18$. At collapse $M_D=0$ and $M_G=M_p$ agreeing with the above figure, however, the horizontal reaction at D is 0.96λ instead of 0.75λ.

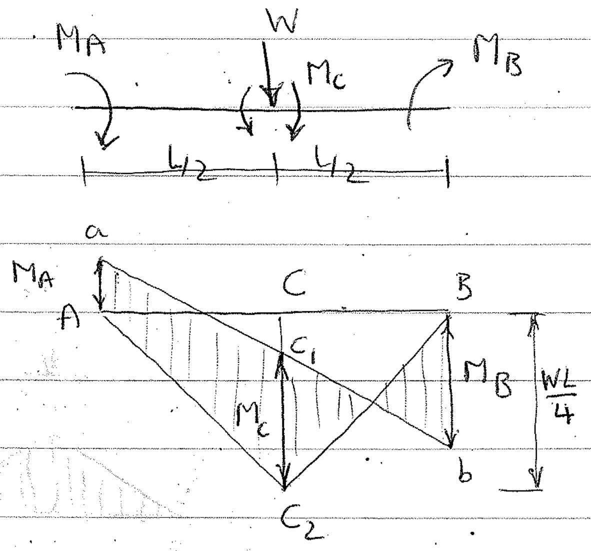
Application : when a new part is added to existing structure or two units attached by a component.

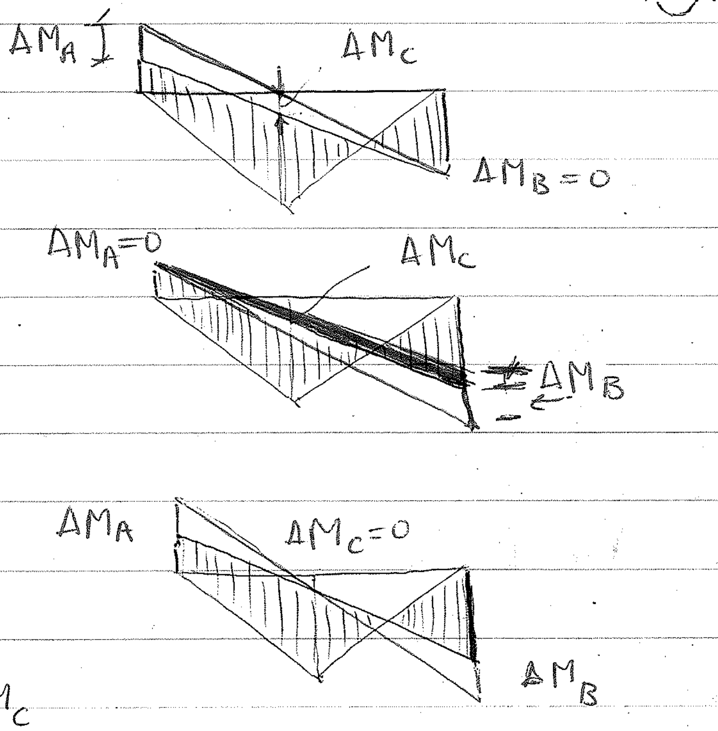
4.7. PLASTIC MOMENT DISTRIBUTION METHOD

The method will first be illustrated by reference to continuous beams and then generalised to other structures.

Consider a centrally loaded (W) beam AB of span L. Denote the bending moment by M_A and M_B, and the central sagging moment M_c.

Allowing for sign convention whereby clockwise moments at A and B are positive, and central moment is + when sagging.



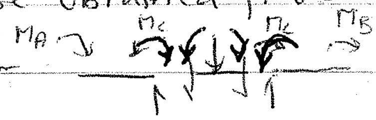


$$\frac{WL}{4} = CC_2 = CC_1 + C_1C_2 = \frac{(M_B - M_A)}{2} + M_c$$

Hence

$$M_A - M_B - 2M_c = -\frac{WL}{2} \qquad (1)$$

Can also be obtained from equilibrium



If change ΔM_A, ΔM_B and ΔM_C are made in M_A, M_B and M_C without change in W, then $(M_A+\Delta M_A) - (M_B+\Delta M_B) - 2(M_c+\Delta M_c) = -\frac{WL}{2}$

(1)&(2) $\Rightarrow$ $\Delta M_A - \Delta M_B - 2\Delta M_C = 0$ (2)

considering the effect of changing only two moments at time,

if $\Delta M_B = 0$ then $\Delta M_C = \frac{1}{2}\Delta M_A$

if $\Delta M_A = 0$ then $\Delta M_c = -\frac{1}{2}\Delta M_B$

if $\Delta M_C = 0$ then $\Delta M_A = \Delta M_B$

These compatible changes are shown in the following operational Table.

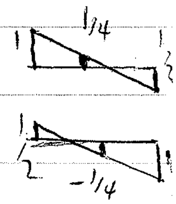
plastic operational Table

operation	ΔM_A	ΔM_C	ΔM_B
1	1	$1/2$	0
2	0	$-\frac{1}{2}$	1
3	1	0	1

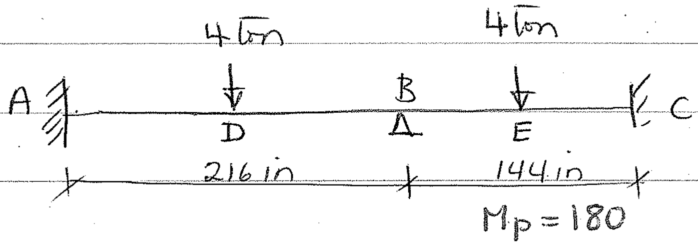
compare this with elastic moment distribution

Elastic operational Table

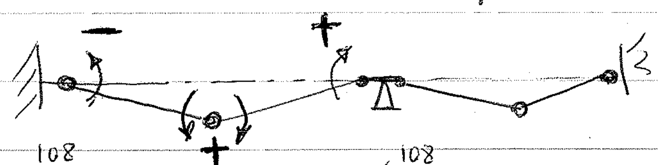
operation	ΔM_A	ΔM_C	ΔM_B
Balance Joint A	1	$1/4$	$1/2$
Balance Joint B	$1/2$	$-1/4$	1



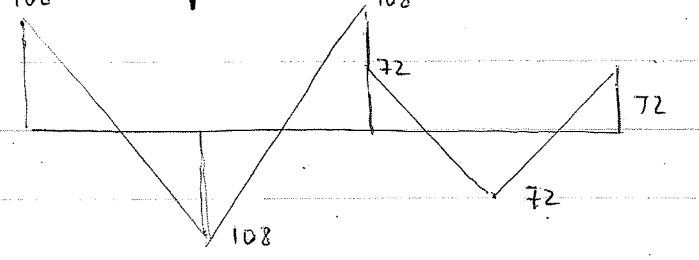
Now consider a two-span continuous beam ABC of uniform cross section for which collapse load factor is to be calculated.



First consider enough plastic hinges to transform the beam (each span) into a mechanism and calculate the bending moment distribution



e.g. $2M_p = \frac{WL}{4} = \frac{4 \times 216}{4} = 216 \quad M_p = 108$



SIGN CONVENTION

An end moment is positive when it acts clockwise on the member, and a centre moment is positive when it is sagging.

A	D	B		E	C
-108	108	108	-72	72	72

A set of equilibrium moments has thus been obtained for each span considered separately. The next stage is to establish equilibrium at joint B. – That is to balance the joint. This will be achieved when the sum of the B.Ms. at the joint is zero.

Using operational table the following 4 cases are possible, if this be done by changing B.M. in one of the beams only.

−108	108	108	−72	72	72
	18 ←	−36			
−108	126	72	−72	72	72

(e)

−108	108	108	−72	72	72
−36 ←		−36			
−144	108	72	−72	72	72

(f)

−108	108	108	−72	72	72
			−36 →	−18	
−108	108	108	−108	54	72

(g)

−108	108	108	−72	72	72
			−36 →		−36
−108	108	108	−108	72	36

(h)

Each of the above B.M. distributions may be tried to obtain a safe estimate of the load factor at collapse for the beam.

For (e) Max B.M. = 126 $\quad \lambda_c \geq \frac{180}{126} = 1.43$

For (f) Max B.M. = 144 $\quad \lambda_c \geq \frac{180}{144} = 1.25$

For (g) Max B.M. = 108 $\quad \lambda_c \geq \frac{180}{108} = 1.67$

For (h) Max B.M. = 108 $\quad \lambda_c \geq \frac{180}{108} = 1.67$ $\quad$ Hence $\lambda_c \geq 1.67$

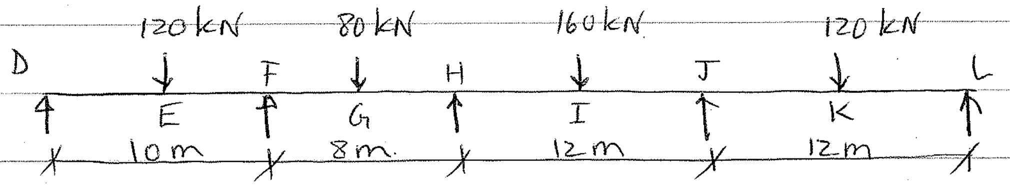
NOTE: The way in which joints are balanced is not automatic as in elastic moment distribution, since the balancing moment may be added to the members meeting at a point in any desired ratio. Moreover, the maintance of equilibrium within a beam by any two of the appropriate operation of the table.

In calculating the collapse loads of frames with uniform full plastic moments throughout, the aim while balancing the joints and carry over the moments, should be to minimize the greatest bending moment, in order to achive greater λ. In structures with variable M_p the aim should be to minimize the greatest ratio of the bending moment to full plastic moment.

EXAMPLE 2:

For the following continous beam it is desired to find suitable plastic moments of resistance such that the beam will just support the given loads at unit load factor.

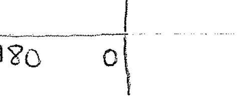
Design



(a)

consider each span completely fixed ended at the state of plastic collapse. Then we will have (as previous example)

$$M_A = M_B = M_C = M_p = \frac{WL}{8}$$



Therefore we have [D-L=H-F-J An arbitray design]

D	E	F	F	G	H	H	I	J	J	K	L
-150	150	150	-80	80	80	-240	240	240	-180	180	180
150 →	75		80 ←		80	80 →		80	-180 ←		-180
			-150 →	-75			-20 ←	+40			
0	225	150	-150	5	160	-160	220	360	-360	180	0

(b)

required M_p 225 160 360 360

Mechanism would not have formed in any of the spans.

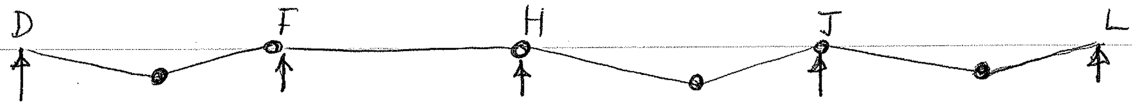
Suppose in design we want to make the first and third spans of minimum possible cross-section. Then we balance as following

[D-F-H-J-L] To minimize 1st & 3rd span

D	E	F	F	G	H	H	I	J	J	K	L
-150	150	150	-80	80	80	-240	240	240	-180	180	180
100 →	50			-80 ←	160				-60 →	-30	
50 →		50	-120 →	-60						90 ←	-180
0	200	200	-200	~~-60~~	240	-240	240	240	-240	240	0

(c)

M_p required 200 240 240 240



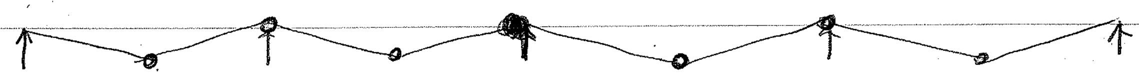
(d)

yet another alternative design is considered in which minimum possible sections are assigned to spans FH and JL. [D-F-H-L-J] to minimize 2nd & 4th span

D	E	F	F	G	H	H	I	J	J	K	L
-150	150	150	-80	80	80	-240	240	240	-180	180	180
	35 ←	-70				160 →	80			60 ←	-120
150 →	75								-60 ←		-60
0	260	80	-80	80	80	-80	320	240	-240	240	0

(e)

M_p required 260 80 320 240



(f)

WEIGHT : Assuming M_p is proportional to the weight of a member

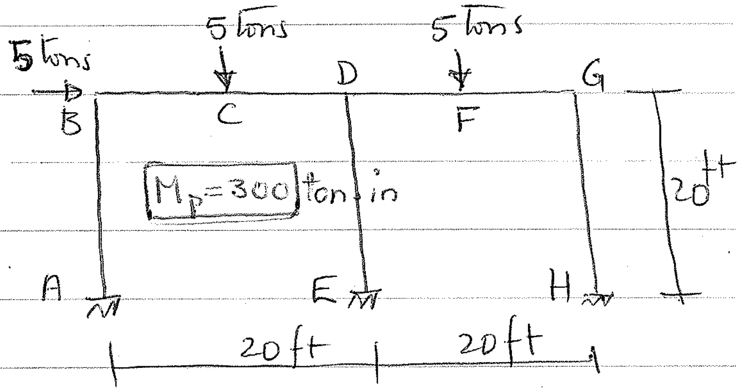
(b) $W = 10 \times 225 + 8 \times 160 + 12 \times 360 + 12 \times 360 = 12,170$

(c) $W = 10 \times 200 + 8 \times 240 + 12 \times 240 + 12 \times 240 = 7,280$

(e) $W = 10 \times 260 + 8 \times 80 + 12 \times 320 + 12 \times 240 = 9,960$

DESIGN OF MULTI STOREY FRAMES BY PLASTIC MOMENT DISTRIBUTION

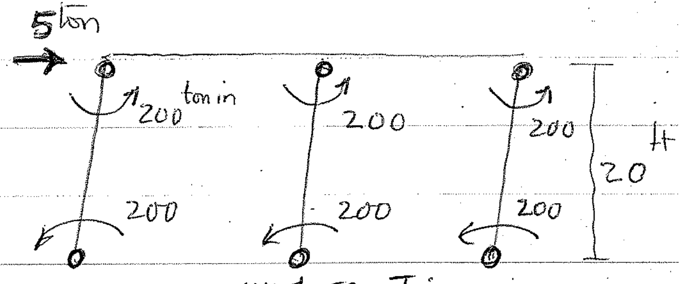
We illustrate the method through the following example. Consider a frame as shown, uniform cross section with $M_p = 300$ ton.in on which effect of axial loads is neglected.



First the joints B, D and G are restrained against rotation, and B.Ms. are calculated from local failures of the beams BD and DG.

In any storey of a multi-storey frame, the sum of the columns moments must equal minus the product of the clockwise storey shear and storey height. In plastic moment distribution, changes of column moments must therefore satisfy the requirement that the sum of the changes within any one storey must be zero.

In our example, equil. with respect to the horizontal load of 5 tons is established by assuming the sway mechanism in fig, giving top and bottom stanchion moments of 200 tons.in.

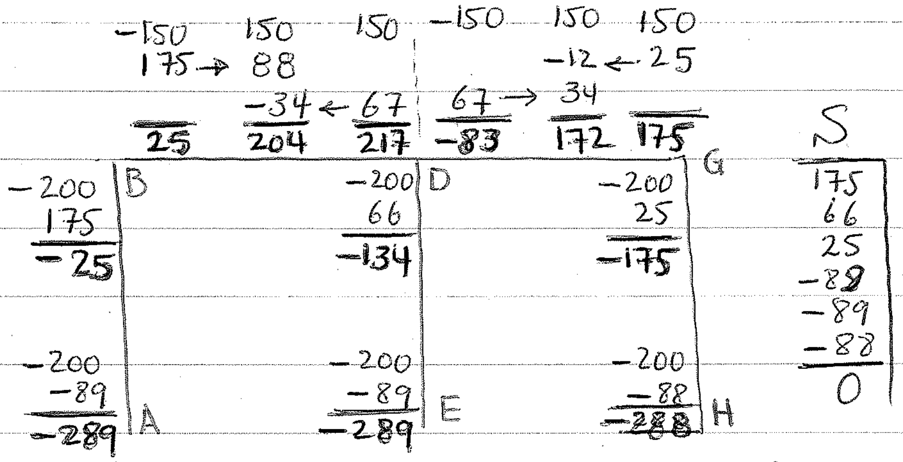


work equation

$5 \times 20 \times 12\,\theta = M_p (6\theta)$

to make inch

$M_p = 200$

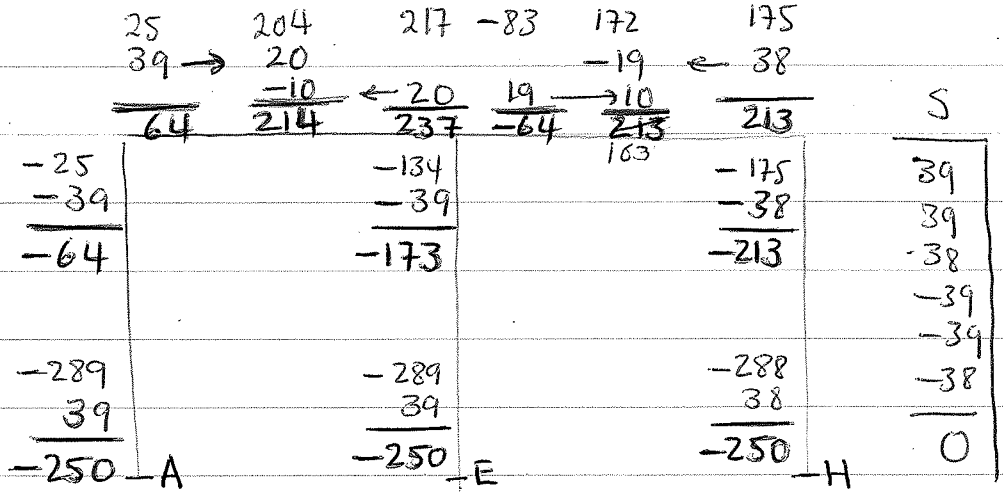


Eqmil. is established at B, D and G by distributing out of balance moments. In the beams, joint balancing moments are carried over to mid-span, thus maintaining eqmil. with respect to the vertical 5 ton loads. To maintain eqmil. with respect to horizontal load, the sum of the top and bottom stanchion moments must remain unchanged. All changes of stanchions moments are collected in the sway column S, so that when horizontal eqmil. has been re-established, the sum of all the bending moments in this column must be zero. **The balancing of the joints B, D and G** introduces moments 175, 66 and 25 into the sway columns, leaving a total of 266 tons in as out-of-balance. This is balanced by distributing it equally between the bases of the stanchions A, E and H.

The total moments are then in complete eqmilibrium with the applied load. The highest bending moment is 289 tons in, so

$$\lambda_c = \frac{300}{289} = 1.04$$

The estimate of λ_c may be improved by decreasing the bending moments at the feet of stanchions.

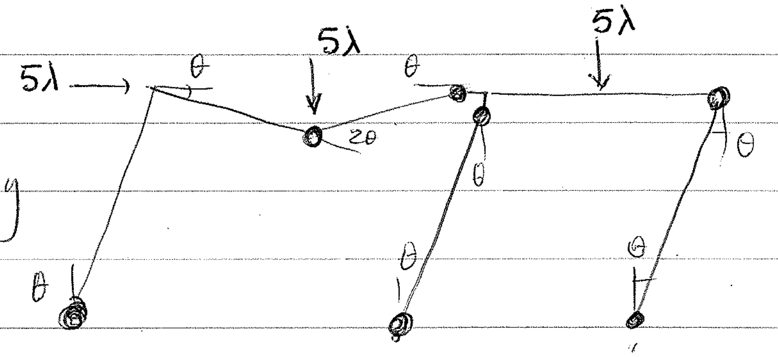


Reduce -289 to -250 Thus add 39, 39, 38 to the feets. To keep the horizontal balance -39, -39, -38 are added to tops of columns. Then Joints are balanced and carried over to the middle of beams. Thus $\lambda_c \geq \frac{300}{250} = 1.2$

The total moment resulting from the above process still fail to represent a state of collapse, since Max B.M. of 250 occurs only at 3 sections A, E and H. The correct value of λ_c could be obtained by further reducing the moments at the feet of the stanchions, and continuing the distribution until the moments did represent a state of collapse. However, the process may be accelerated by using the derived bending moments to deduce the probable collapse mechanism. In present case

Since frame is uniform, it is only necessary to number the bending moments in descending order of magnitude until sufficient hinge positions have been found to transform it into a mechanism.

Introducing 7 plastic hinges in selected sections the following mechanism is obtained



and the corresponding collapse equation is

$$5\lambda \times 240\theta + 5\lambda \times 120\theta + 5\lambda \times 0 = 8 \times 300\theta$$

$$\lambda_c = 1.33$$

This is obviously an upper bound to λ_c

Thus

$$\frac{300}{250} = 1.200 \leq \lambda_c \leq 1.333$$

Now B.Ms are modified to correspond to the mode of collapse given in above figure. The plastic hinge value becomes

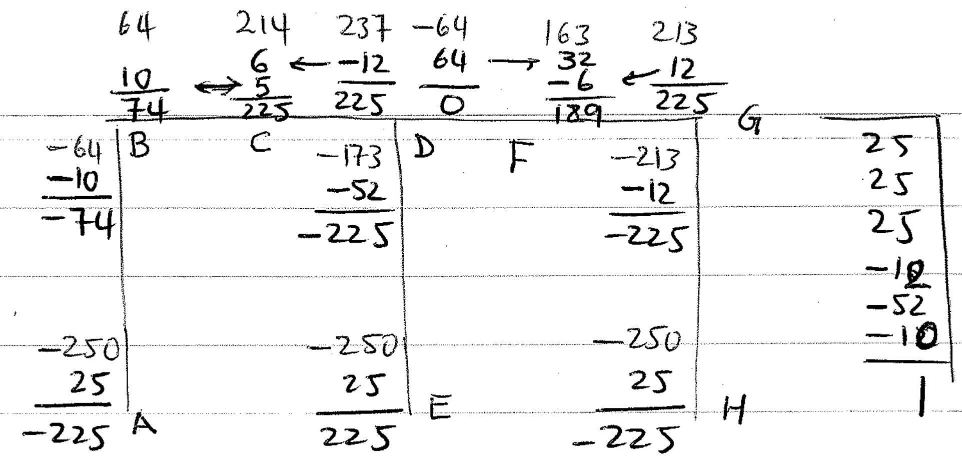
$$300/1.333 = 225 \text{ ton.in}$$

Thus all moment in the assumed hinge positions are modified to agree with this value. (see next page)

Since B.M. nowhere exceeds 225, Thus the assumed mode of collapse is a correct one and $\lambda_c = 1.333$.

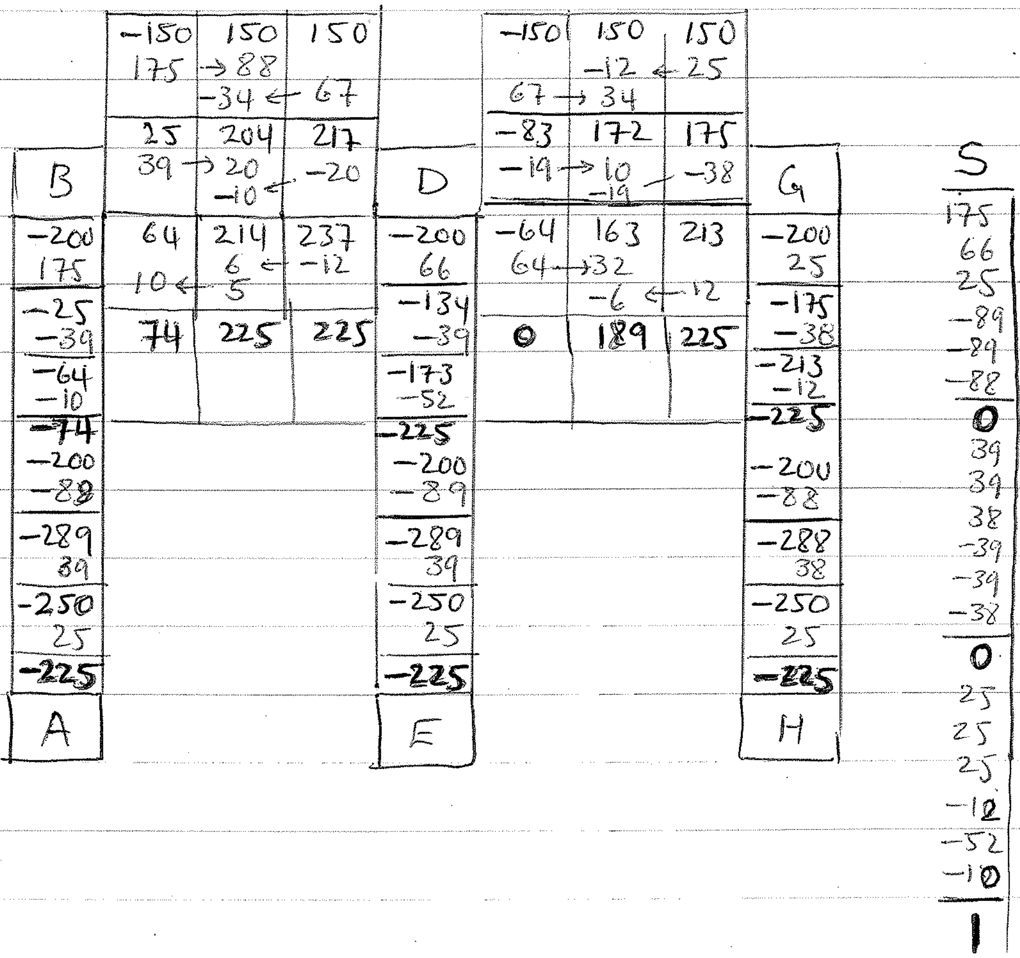
A → E → H → G → D → B → G → B → C → D → F

Shear Balance Balance Balance needed for 225 Balance Finish



Final B.M. Distribution

The above calculation can be presented in a more compact table as follows:



692 kN m. The central moment is reduced by approximately this amount (always err on the safe side and use a slightly enhanced value, in this case, say, 30 kN m). The balance is carried to end A, necessitating a change in the column moment at A of 60 kN m. Horizontal equilibrium is restored by adding −60 kN m to the column moment at C, as shown in fig. 2.15(f).

Similarly case 2 in Table 2.1 is applied to span CD, revealing a maximum span moment of 1843 kN, 7 kN m above the plastic moment of 1836 kN m. The central moment is reduced by, say, 10 kN m and the balance carried to end C, where joint equilibrium is restored by adding (60 + 20) = 80 kN m to the moment beam BC. Finally, balance is restored in beam BC by carry-over to the centre (see fig. 2.15(f)).

A final check of maximum sagging moments in the beams shows the following values compared with the respective plastic moments (in parentheses): AB 686 (692), BC 757 (959), CD 1834 (1836). Hence the design is safe.

For the sake of clarity, the steps in figs. 2.15(d)–(f) have been shown on separate tabulations, but in practice all calculations could be carried out in the same tabulation (see fig. 2.15(g)). It is, of course, necessary to check also the ability of the frame to withstand the load situation when the maximum load is applied to the cantilever DE. Changes of moment which will accommodate the increase of cantilever moment at D and the decrease in plastic moment of column D to 553 kN m are shown in italics in fig. 2.15(f) or fig. 2.15(g), and it is readily seen that a state of equilibrium is achieved in which the plastic moment of resistance is nowhere exceeded. Hence the frame will safely support this modified loading.

While the design which has been obtained is a safe one, it would not necessarily be adopted, since the convenience of fewer changes in beam section would have to be considered. Plastic moment distribution gives a ready means of checking such designs, and also of exploring solutions in which changes of beam section are made within the spans rather than at the ends.

2.6.2 Design of Multi-storey Frames

In any storey of a multi-storey frame (fig. 2.16), the sum of the column moments ($M_{12A} + M_{21A} + M_{12B} + \ldots$) must equal *minus* the product of the clockwise storey shear Q_{12} and storey height h_{12}. In plastic moment distribution, *changes* of column moment must therefore satisfy the requirement that the sum of the changes within any one storey must be zero.

Suppose it is required to design by plastic moment distribution the frame in fig. 2.17, loaded as shown (factored loads). It is also supposed that the available sections for the beams and columns have the plastic moments of resistance indicated, the effect of axial loads on the plastic moments of the columns being ignored in this example. Considering all the joints as being fixed against rotation (but not against horizontal movement), moments to produce equilibrium in the beams (moments of magnitude $WL/8$), and in the columns (where it

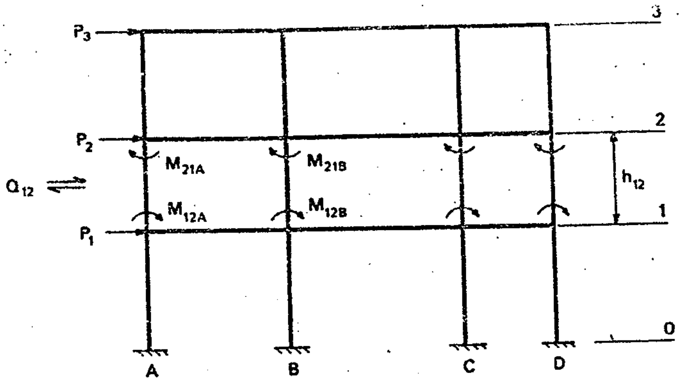
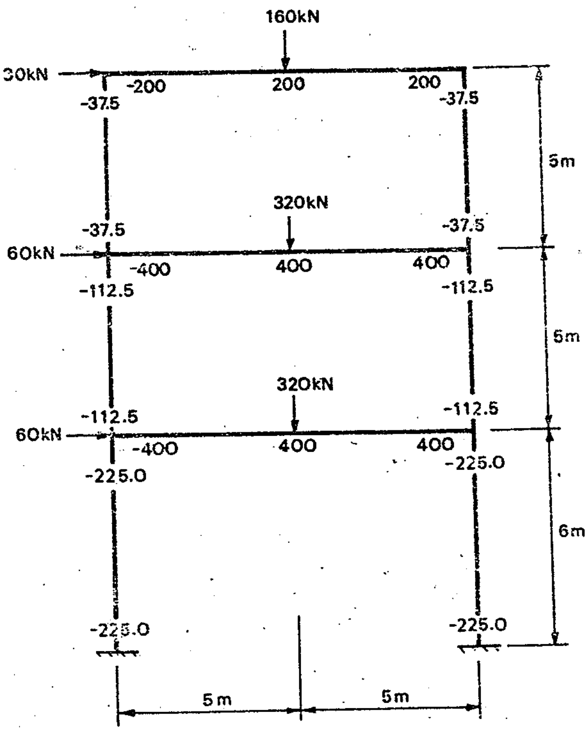


Fig. 2.16

is assumed that the wind shear is divided equally between the columns in any one storey) are given in fig. 2.17.

(1) Figure 2.18(b) shows the derivation of one possible set of sections, chosen to given the minimum possible column sections. The absolute minimum required



$-2(2M) = 30 \times 5$
$M = -37.5$

$-2(2M) = 90 \times 5$
$M = -112.5$

$-2(2M) = 150 \times$
$M = -225.0$

Plastic moments of available sections
Beams: 180, 210, 250, 300, 360, 420, 500, 580, 660, 750, 850 kN m
Columns: 45, 70, 105, 135, 160, 200, 250, 320, 400, 480, 560, 660 kN m

Fig. 2.17

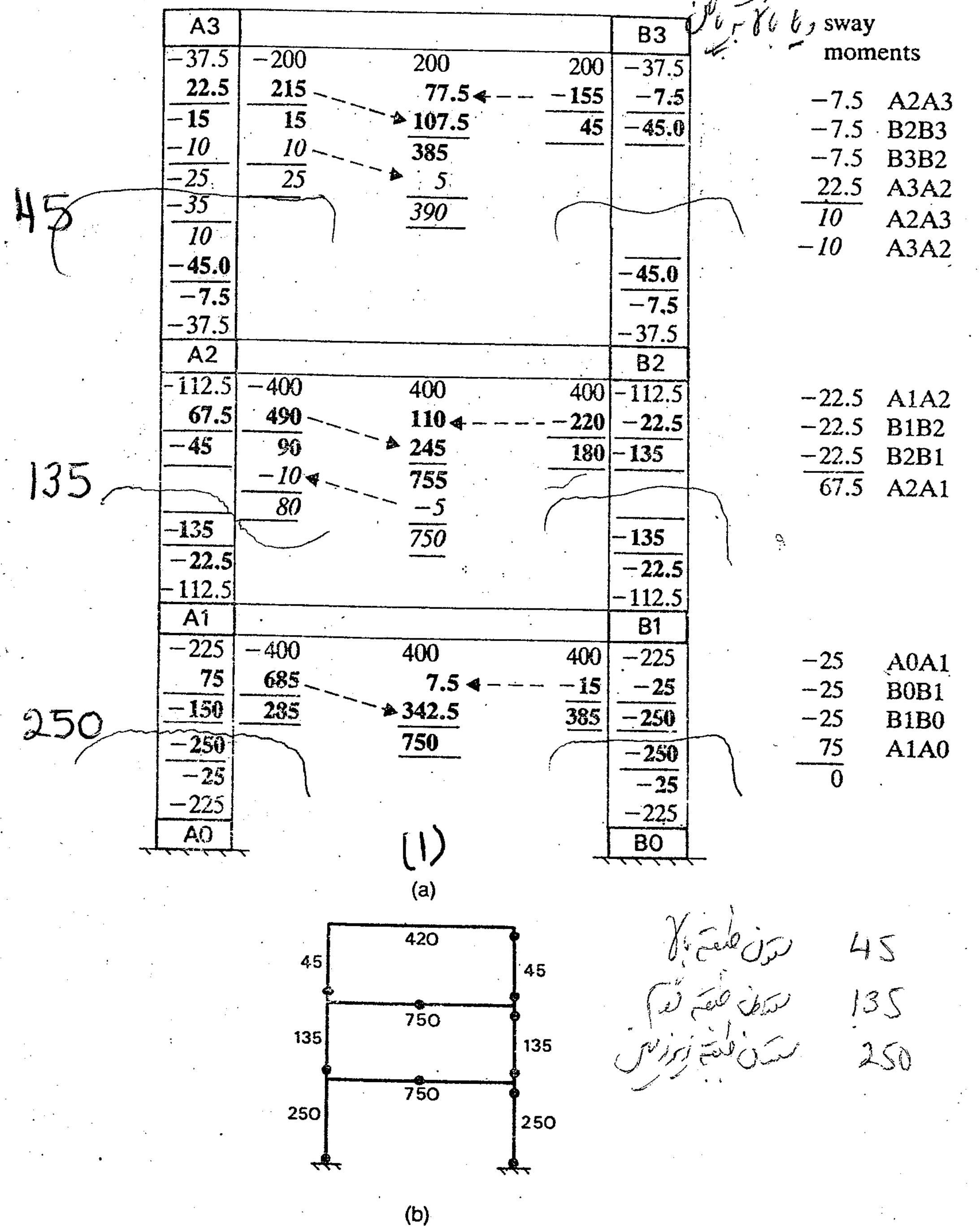


Fig. 2.18 Minimum column design. (a) moment distribution; (b) design sections and hinge pattern

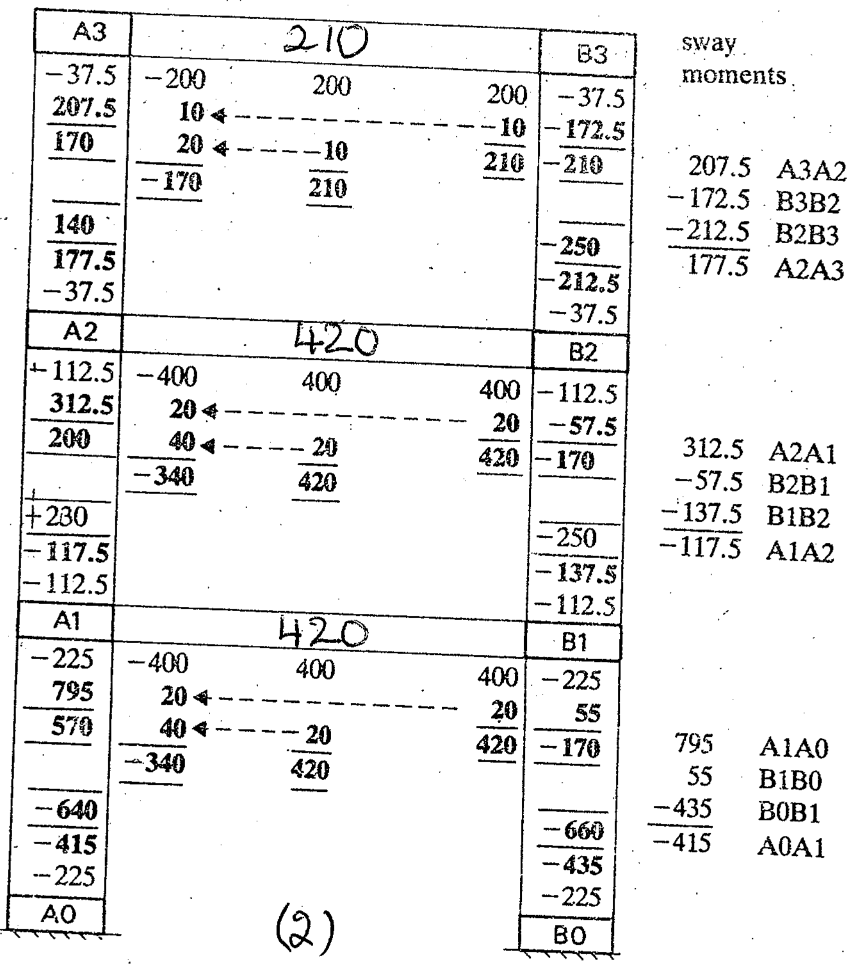
plastic moment of resistance of the columns in the top storey being 37.5 kN m, the lowest available section (M_p = 45 kN m) is chosen. Plastic hinge moments are assumed in the top storey columns at A2, B2 and B3 but not at A3, in order to help towards the balancing of this joint. The changes in column moments (3 × (−7.5)) are taken to the 'sway moments' array. The sum of all column moments, both top and bottom of each column, within any one storey, when added to the array for each storey, must be zero to maintain equilibrium, and to preserve this equilibrium, a moment of 22.5 kN m is added to column A2A3 at A3. The joints are balanced by adjusting the adjoining beam moments, and then carrying moments over to mid-span.

A procedure similar to the above is followed in the first- and ground-floor storeys, using column sections with plastic moments of resistance of 135 and 250 kN m respectively. The maximum beam moments, after balance has been achieved, are found to be 750 kN m in beam A1B1, 755 kN m in beam A2B2 and 385 kN m in beam A3B3. Since a beam section with a plastic moment of resistance of 750 kN m is available, means are explored of reducing slightly the maximum moment in beam A2B2, and the changes of moment shown in italics in fig. 2.18(a) are made. In so doing, it is borne in mind that plastic hinges are most likely to form on the leeward side of a structure under the influence of wind loading. It is not found possible to reduce the plastic moments required in the other two beams, and the design plastic moments are summarised in fig. 2.18(b).

(2) An alternative design based on minimum beam sections as opposed to the assumption of minimum column sections is shown in fig. 2.19(a). The absolute minimum beam moments are 200 kN m in the roof beam and 400 kN m in the other two beams, leading to the choice of beam plastic moments of resistance of 210 and 420 kN m respectively. In accordance with the usual pattern of beam plastic hinges in frames under vertical loading combined with horizontal loads acting towards the right-hand side, plastic hinge moments are assumed to occur at mid-span and at the right-hand ends of the beams. Columns with plastic hinge moments of 250 kN m are found to be adequate in the upper two storeys, but at ground level sway equilibrium requires the use of columns having plastic moments of 660 kN m (fig. 2.19(b)).

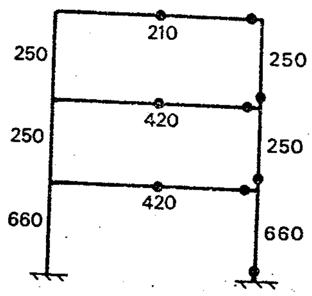
(3) In practical design, stability considerations, as well as the effect of axial loads on plastic moment values, will affect the choice of column sections. Suppose that, in the present example, stability considerations lead to minimum column sections having plastic moments of 250, 135 and 70 kN m respectively in the ground, first- and top-storey levels respectively. A convenient choice of column sections might then be as shown in fig. 2.20(b), the column splice being above joints A1 and B1 at a sufficient height for full benefit to be taken of the 250 kN m plastic moment of resistance at the bottom of column lengths A1A2 and B1B2.

The beam sections will be intermediate between the beam sections in figs. 2.18(b) and 2.19(b). Taking plastic hinge moments of 660 kN m occurring at mid-span of beams A1B1 and A2B2 and working up from the bottom of the



(a)

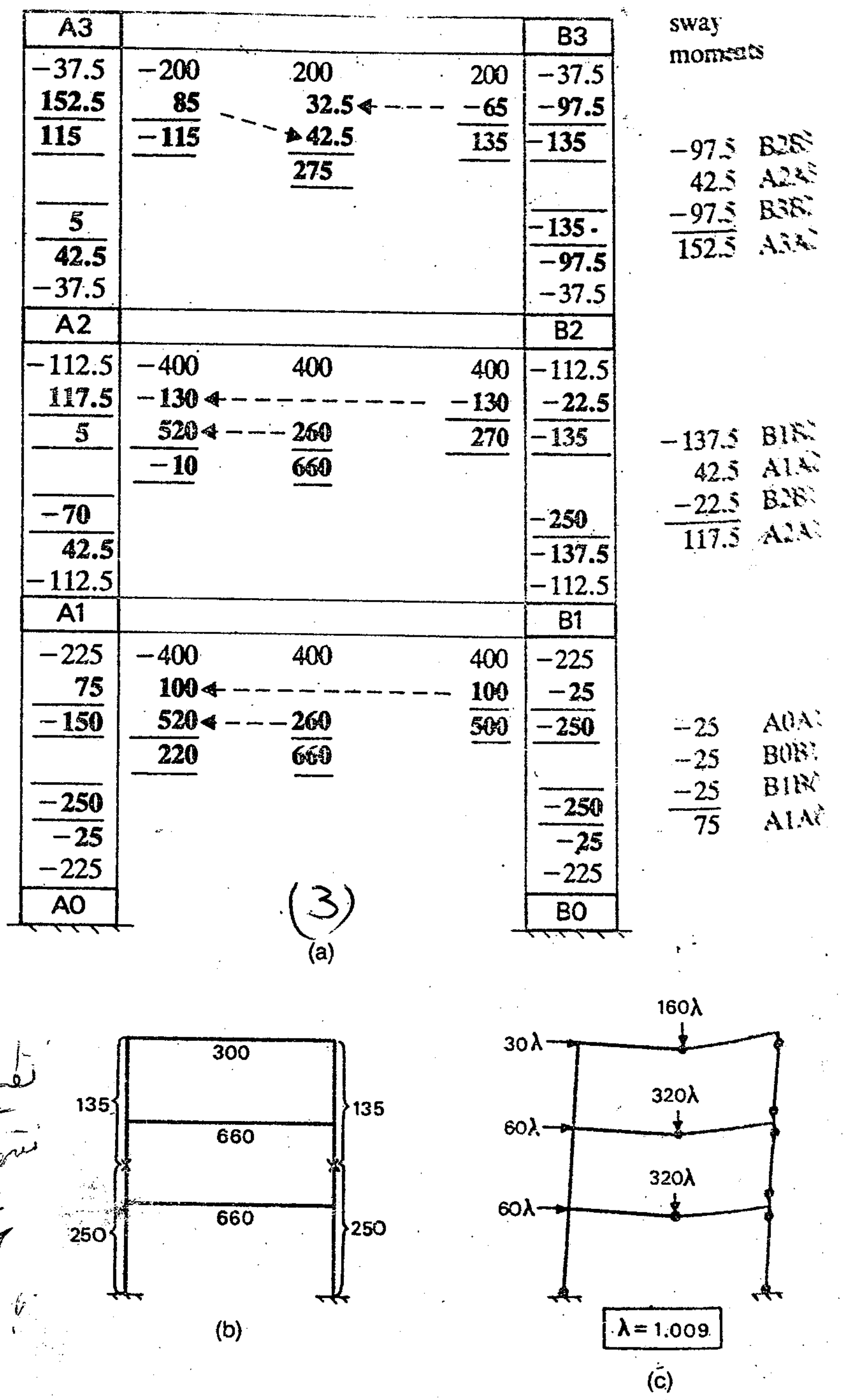
Combined



(b)

A3B3	210
A2B2	420
A1B1	420

Fig. 2.19 Minimum beam design. (a) moment distribution; (b) design sections and hinge pattern



135
250

Fig. 2.20 **Final design.** (a) moment distribution; (b) design sections; (c) collapse mode.

frame, the bending moments shown in fig. 2.20(a) are derived. The sagging moment in the roof beam is found to be 275 kN m, thus requiring the section with the plastic moment of resistance of 300 kN m.

It will be appreciated that the steps taken in plastic moment distribution are not unique, and skill in the use of the method comes only with practice. It is useful to have in mind the likely form of the plastic hinge mechanism—for example, the design in fig. 2.20(b) has the collapse mechanism shown in fig. 2.20(c). The mid-span moment in the roof beam in fig. 2.20(a) is not quite equal to the plastic hinge value and a work equation shows that, for this collapse mechanism to take place, the loads must be 0.9% above those in fig. 2.17.

2.6.3 Design Example of Four-storey Frame

The rigid frame shown in fig. 2.21(a) is to be designed by plastic moment

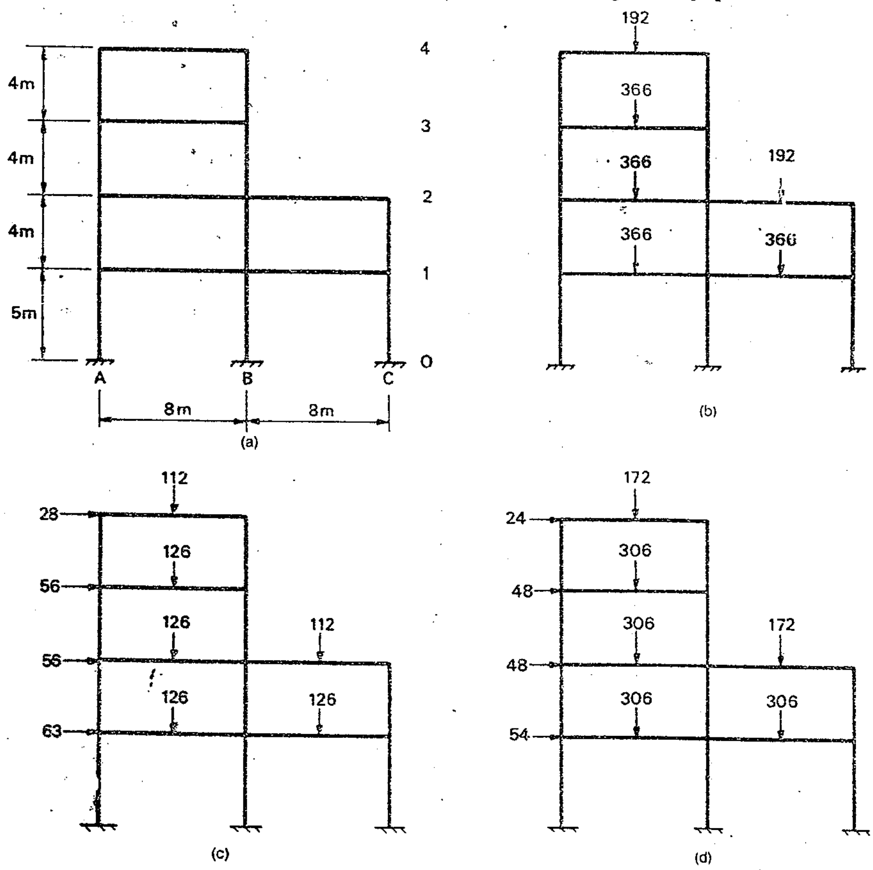


Fig. 2.21 (a) Dimensions; (b) equivalent factored dead plus imposed vertical loads (kN); (c) equivalent factored dead plus wind loads (kN); (d) equivalent factored dead plus imposed plus wind loads (kN)

4.8 Upper and lower bounds

The safe and unsafe theorems are most effective as an aid to the calculation of collapse loads when they are used alternately. It is then possible, as the analysis proceeds, to place progressively narrow limits on the load factor at collapse. In extensive frames, the methods of combination of elementary mechanisms and plastic moment distribution are the most convenient means of applying the minimum and max principle (safe and unsafe theorems), and it is advantages to use both methods of analysis for the same problem. The elementary mechanisms with the lowest load factors are first combined, giving an upper bound on the collapse load. plastic moment distribution is then used to derive bending moments appropriate to the tentative collapse mechanism, thereby giving a lower bound. If the upper and lower bounds are not identical, the bending moment distribution will, by the existance of bending moments in excess of the full plastic values, reveal which further combinations of elementary mechanisms should be tried. The use of the two methods in conjunction avoids the necessity of trying a large number of possible combinations of mechanisms when the method of combining elementary mechanisms is used alone. It also provides a check on the working, since the bending moments at the hinge positions are calculated by two independent means.

For moderate complexity guess the collapse mode by considering an arbitrary mechanism X_1 and writing virtual work equation a load factor λ_1 is obtained where

$$\lambda_1 \geq \lambda_c$$

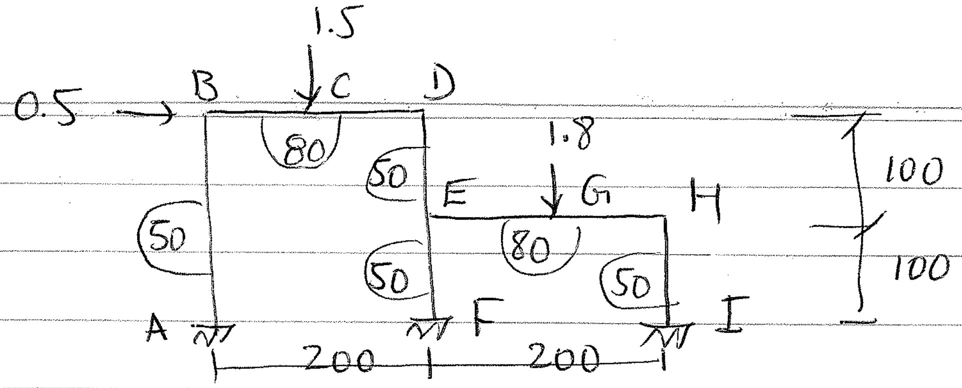
Let the BM corresponding to X_1 be determined and denote by M_1.

Unless $\lambda_1 = \lambda_c$, the BM M_1 will exceed the M_p over some part of the structure. Reduce these BM by multiplying by some common factor μ_1 such that BM now nowhere exceeds M_p. The resulting BM will be in equilibrium with loads of load factor $\mu_1 \lambda_1$. Hence

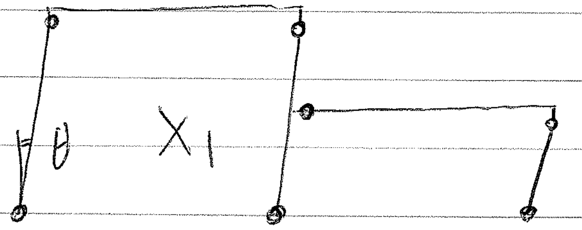
$$\mu_1 \lambda_1 \leq \lambda_c \leq \lambda_1.$$

When one mechanism X_1 has been postulated for any structure, and the corresponding BM M_1 is calculated, it is always possible (unless $\mu_1 = 1$) to deduce at least one mech. X_2 which gives a load factor λ_2 lower than λ_1. A mech. X_2 may be found by selecting hinge positions from those of mech. X_1, and the sections where $M_1 \geqslant M_p$, provided the sign of the bending moment M_1 corresponds to the direction of rotation of the hinge, and provided there is at least one hinge in mech. X_2 at which $M_1 > M_p$. This statement follows immediately from the virtual work equations obtained by considering mech. X_2, and may be used to decrease systemmatically the upper bound on λ_c until the correct value is obtained. Corresponding to the upper bound λ_2 obtained from X_2, there will be a revised lower bound $\omega_2 \lambda_2$. While it is <u>not</u> always true that $\omega_2 \lambda_2 > \omega_1 \lambda_1$, this is usually found to be the case. Hence the lower bound on the collapse load is progressively increased as the upper bound is decreased.

Example is given in the following:



Consider a sway mech. as follows:

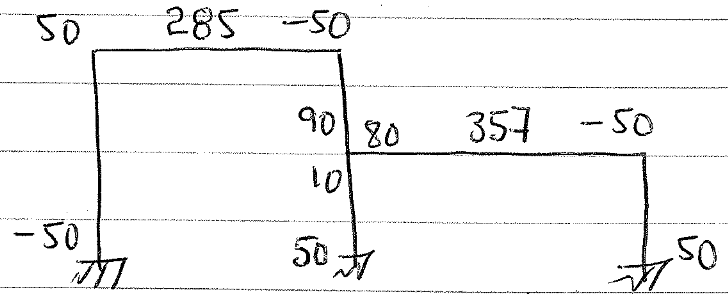


$2 \times 50\,\theta + 2 \times 50\,\theta + 2 \times 50\,\theta + 80\,\theta$

$= 0.5 \times 200\,\theta$

$\boxed{3.80 = \lambda_1}$

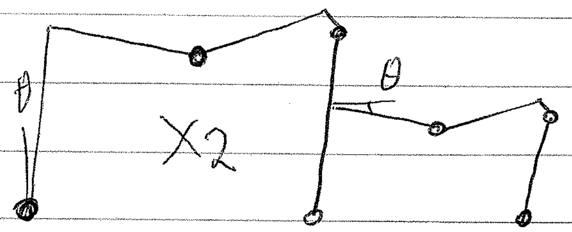
the corresponding B.M. is as follows

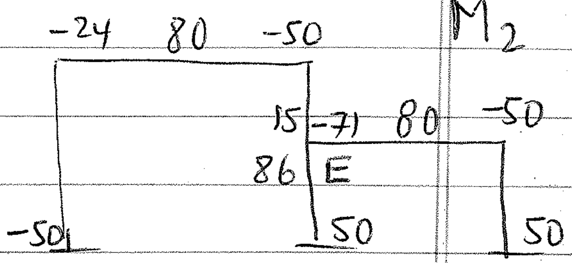


the highest ratio is $\frac{357}{80} = 4.46$ Hence $\frac{3.80}{4.46} = 0.85$

Therefore $\boxed{0.85 \leq \lambda_c \leq 3.80}$

Obviously plastic hinges should be inserted at C and G. These hinges replace hinges at B and E





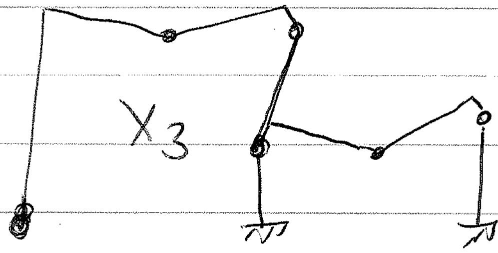
leading to $\lambda_2 = 1.56$. Hence $1.56 \times \frac{50}{86} = 0.91$ and

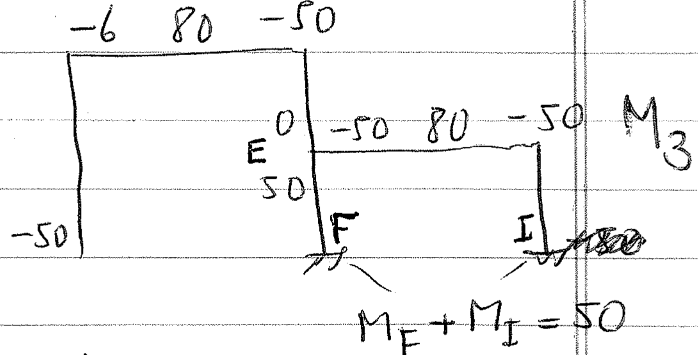
$\boxed{0.91 \leq \lambda_c \leq 1.56}$

OR

$\frac{86}{50} = 1.72 \quad \frac{1.56}{1.72} = 0.91$

Only at E yield is violated. Hence a hinge is inserted at E as follows:





Corresponding to $\lambda_3 = 1.44$ and corresponding BM is within the limits. M_F and M_I are not known (indeterminate), however, for equilibrium $M_F + M_I = 50$ so yield is satisfied and

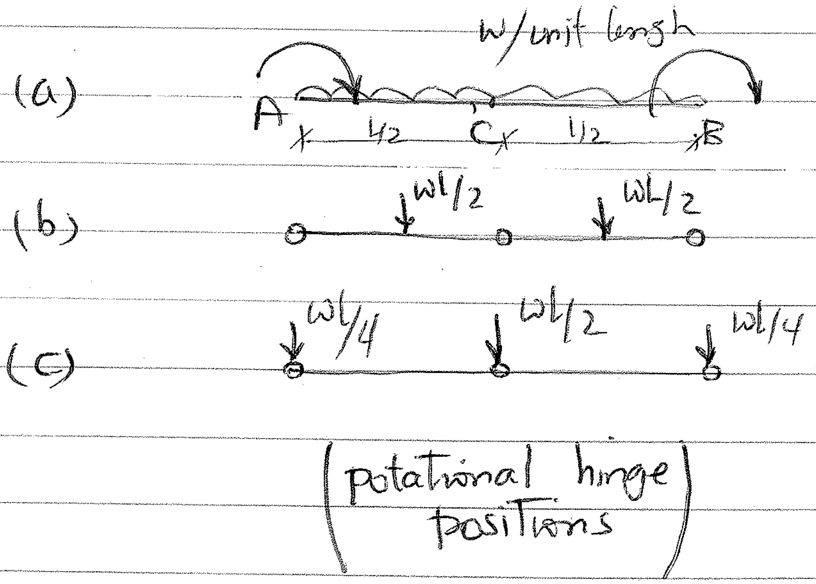
$$\boxed{\lambda_c = 1.44}.$$

Uniformaly Distributed Loads

a) General method

When distributed loads are present, the number of possible hinge positions become infinite. In the first stages of an analysis it is best to assume plastic hinges at a limited number of positions. The solution obtained with simplifying assumptions may then be modified to allow for the possible occurance of plastic hinges at other cross sections of the structure.

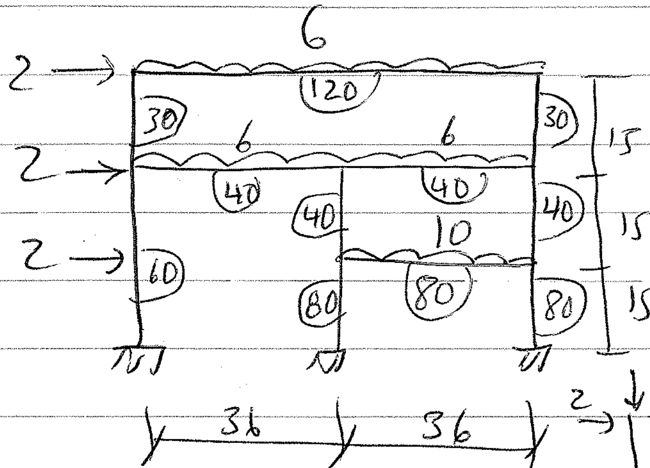
In the primary analysis hinges are consider at member ends and the center (in pitched roof frames at the centers of rafters).



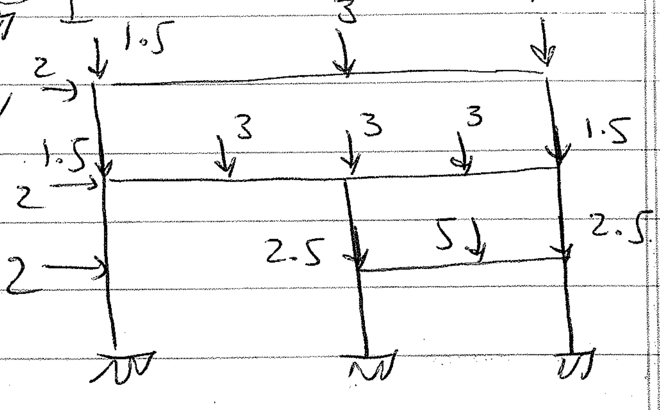
The effect of distributed loads on the collapse mechanism is to move the "sagging moment" hinges in the beams. It is rarely that distributed loads affect the basic mode of collapse.

The load factor based on equivalent concentrated loads, λ_0, is necessarily greater than the true load factor at collapse, λ_c, since it is based on a mechanism differing from the correct mechanism. However, this difference is usually small.

For figures (a) and (b) we obtain



(a)

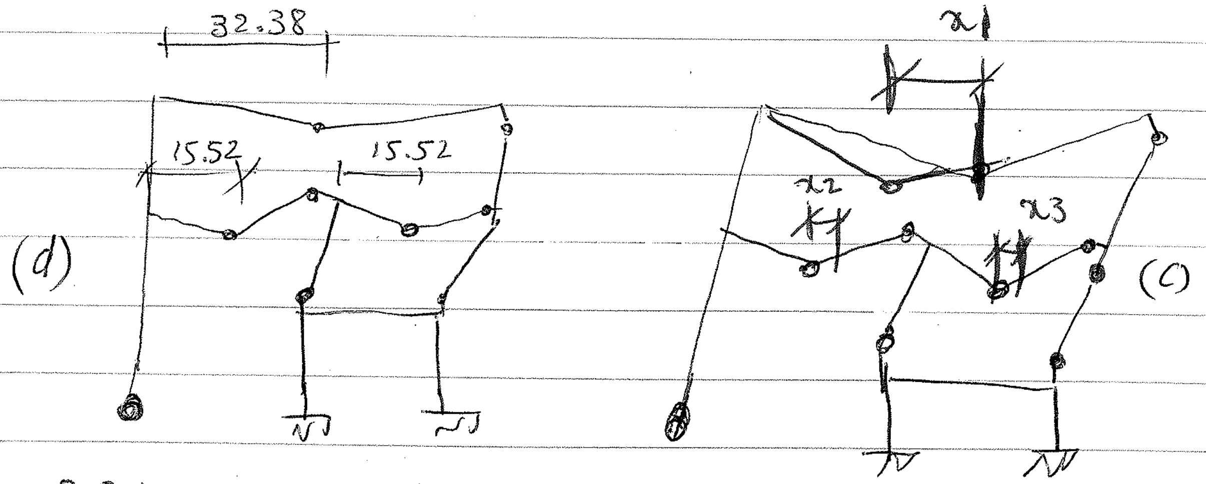


(b)

For collapse mode (c) using V.W. one obtains

$\lambda = 2.311$ (NO allowence)

With proper allowance for distributed load we have $\lambda_c = 2.290$ (with allowence)



$\lambda = 2.290$

The difference is 0.9% overestimate in the collapse load factor.

further reduction:

for $x_1 = 3.64$ $x_2 = x_3 = 2.54$

$\lambda = 2.2899$

for $x_1 = 3.62$ $x_2 = x_3 = 2.48$

$\lambda = 2.2895$

Lower bound can be stablished as $\lambda = 2.242$

Even with no effort

$2.242 \leq \lambda \leq 2.311$

and the difference is only 3%