Plastic analysis and design of frame structures
By: A. Kaveh

Contents

References

1. Baker, JF, Horne, MR, and Hyman, J., The Steel Skeleton, Vols. 1 and 2, Cambridge University Press, 1954 and 1956.
2. Beedle, LS., Plastic Design of Steel Frames, John Wiley, 1958.
3. **Baker, JF, and Hyman, J., Plastic Design of Frames, Vols. 1** and 2, **Cambridge University Press, 1969.**
4. Majid, KI., Non-linear Structures, Butterworth, 1972.
5. Horne, MR., Plastic Theory of Structures, Pergamon Press, 1979.
6. Horn MR. and Morris, L.J., Plastic Design of Low-rise Frames, Williams Collins Sons, 1981.
7. Neal, BG., The Plastic Methods of Structural Analysis, Chapman and Hall, 1985.
8. Bruneau, M., Uang, CM and Whillaker, A., Ductile Design of Steel Structures, McGraw-Hill 1998.
9. David, A. Nethercot, Limit State Design of Structural Steelwork, Spon Press, 2001.
10. Moy, Stuart SJ., Plastic Design of Concrete and Steel Structures, Wiley, 1981.
11. Nethercot, DA., Limit State Analysis of Structural Steel Work, Spon Press, 3rd edition, 2001.

Other References: Maier-Leibnitz, Symonds, Grigorian, Munro, Smith, Watwood, Deeks, Kaveh, Mokhtar-zadeh, Khanlari, Jahanshahi, Harrison, Morice and others.

Chapter 1: Basic concepts and definitions

Chapter 2: Simple beams and frame; derivation of plastic load factor

Chapter 3: Fundamental theorems of plastic analysis and design

Chapter 4: General methods of analysis and design

Chapter 5: Non-conventional structures

Introduction
The Grillages
The vierendeel girders
The arches

Chapter 6: Deflections and stability

Introduction
Calculation of deflections at the point of collapse
The effect of deflection on the collapse load
Concluding remarks

Chapter 7: Assumptions and aims of the plastic design

Introduction
Definitions, assumptions and concepts used in the plastic analysis and design

Chapter 8: Topological formulation of plastic analysis and design

Introduction
Mesh and nodal description
Cycle basis for and efficient analysis and design
Duality
Plastic limit synthesis

Chapter 9: Heuristic approaches for optimal plastic analysis and design

Introduction
Formulation for optimization
Recent developments in optimal plastic analysis and design using heuristic algorithms

Chapter 1

INTRODUCTION

The cinema is responsible for the interest which has grown in Britain since 1936 in the possibility of basing a design method on the plastic behaviour of structures. It was neither a frank documentary nor a stirring drama set among the sky-scrapers which turned attention from the elastic to the plastic range, but the riveted steel framework of a cinema under erection in Bristol at the time the Steel Structures Research Committee's *Recommendations for Design* were being drafted. It will be remembered, from the account given in Volume I, that this rational design method based on the elastic behaviour of framed structures could deal only with the steel skeleton in its most regular form of lines of vertical stanchions to which horizontal beams are attached. The investigators who had carried the main burden of the Committee's work realized, therefore, as they passed the site of the cinema day after day and saw its framework rise, the sweeping curve of the gallery girder and the light but complex structure of the projection room, that the *Recommendations* would apply to only a small part of this particular building. What is more, they knew from the tests on the three existing buildings, the hotel, the office and the residential flats, which they had carried out (Vol. I, Chap. 7), that the elastic behaviour of a cinema frame would be so complex and variable that no practicable general rational method of design of the Steel Structures Research Committee's type could be derived. They were equally confident that the engineers responsible for this particular structure had been dissatisfied with many of the assumptions and approximations forced on them by the orthodox method of design and would have been nonplussed by any change in method of fabrication, such as the use of welded connexions then becoming possible, which might alter the character of the structure.

This was an intolerable position. It was hard to believe that no better general tool could be devised than the orthodox method of the Code of Practice(1.1) described in Volume I, Chapter 2. If this were so then the riveted and bolted structure had reached its final and most economical form and the community would be denied indefinitely the full benefit of the rapidly developing technique of joining members together by welding. Might it not be that the whole basis of elastic design was at fault and that the path which designers had been following for nearly a century was nothing more than a blind alley?

1

BASIC Concepts and definitions

1. Basic Concepts and definitions

1.1. Introduction

The Cinema under erection in Bristol, where the "Steel Structures Research committee's Recommendation for design" were being held, is responsible for the interest which has grown in Britain in 1936 in the possibility of basing a design method on the plastic behaviour of the structures. The committee members passed the site of the cinema day after day and saw its framework rise, the sweeping curve of gallery girder and the light box complex structure of the projection room. Their recommendations would apply to only a small part of this particular building. Thus a new basis for design was inavoidable.

1.1.2 THE WORK OF MAIER-LEIBNITZ

Fortunately at about this time the investigators learnt of the work of prof. H. Maier Leibnitz of The Technische Hochschule, Stuttgart

(1929) Versuche mit eingespannten und einfachen Balken von I-Form aus St. 37, Bautechnik, 7, 313.

who had loaded encastre' and continuous beams carrying them out of the elastic into the plastic range. He had taken a 40X40 cm I beam 4m long and placed it on simple supports A and B as shown in Fig. 1.1. Loads W applied at the third points of the span and loads P at the ends so adjusted that the cross-sections cross-section of the beam over the supports remained vertical as the loads W were increased.

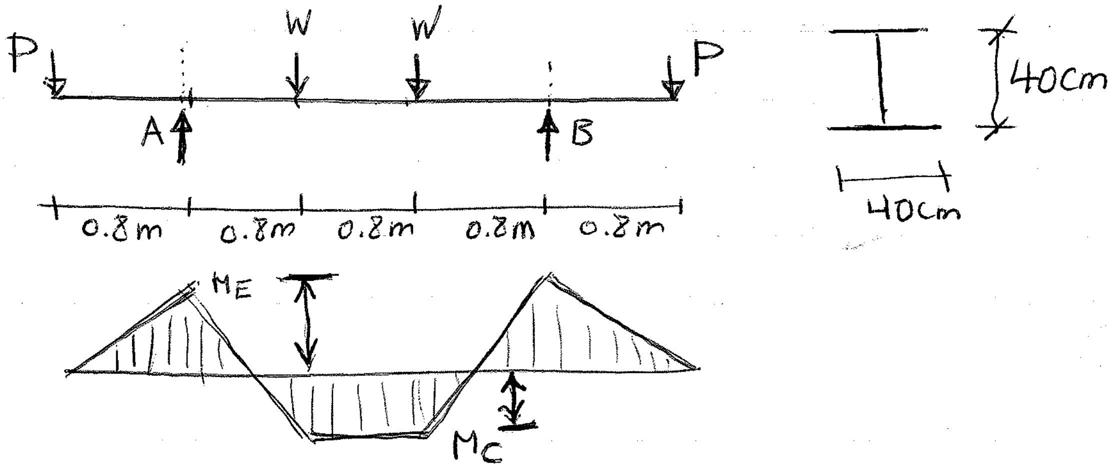


Fig. 1.1. Maier Leibnitz beam representing a clamped beam at two points A and B

Therefore AB behaved as an encastre' beam. With W and P being known M_E and M_C could easily be deduced. Then deflection at mid-span and moment M_E and M_C were plotted against W as in Fig. 1.2 and Fig. 1.3.

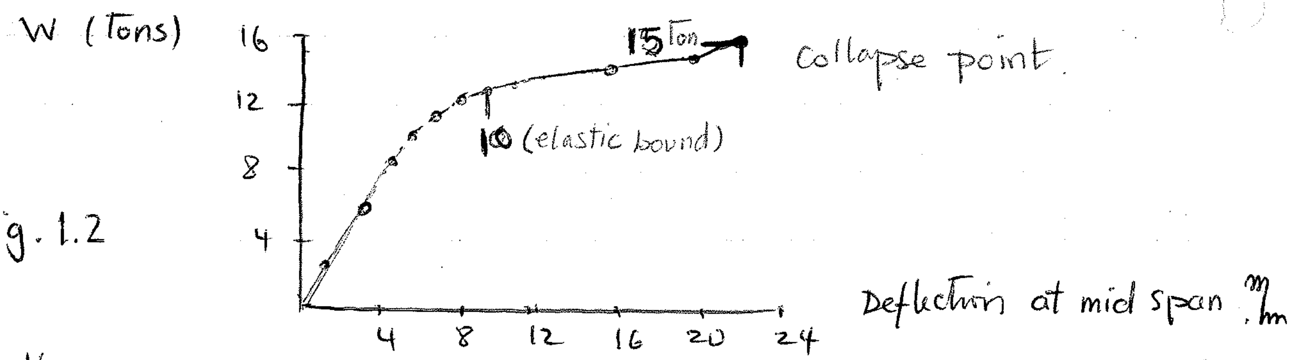


Fig. 1.2

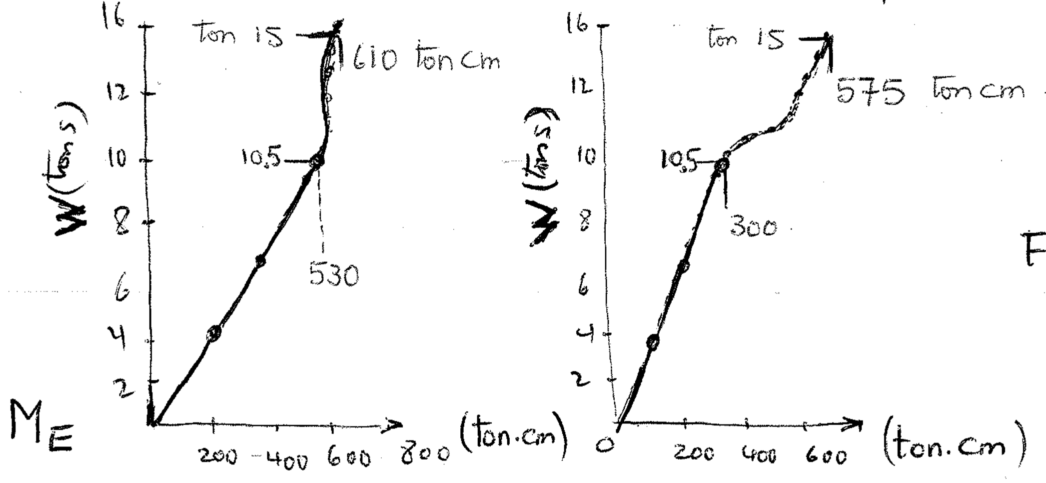
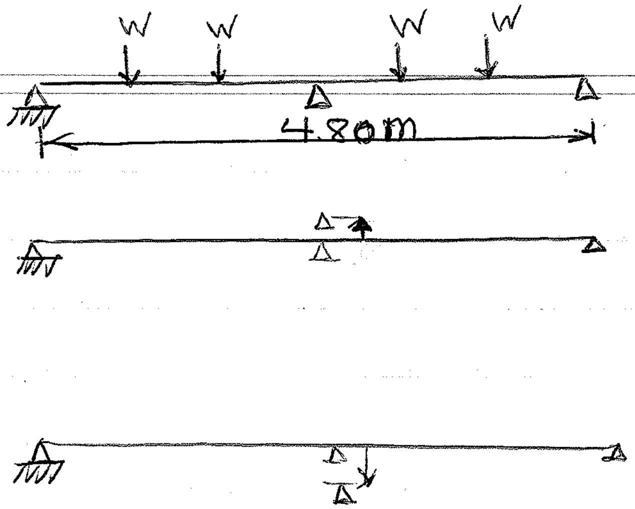


Fig. 1.3

~~The beam behaved elasticdlly until W reached almost~~ 10 tons. When W was 10.5 ton the end moments had the values 530 and the centre moment had the value 300 tons.cm. After that as the load was increased further, M_E remained sensibly constant while M_c continued to increase but at an accelerated rate, until the deflection having become very large. The beam collapsed when W = 15 ton, $M_E = 610$ and $M_c = 575$ cm. Tons.

Therefore this fixed ended I beam supported a load more than 1.5 times that which produced the first yield. Moreover, it appeared that the end moments, which in the elastic range would have been twice the center moment, having reached certain value, remained almost constant while the centre moment increased until it, in its turn, reached approximately the same critical value and collapse occured. This state of affairs suggested the formation of plastic hinges at A and B, allowing free rotation while providing a constant restraining moment. Then a plastic hinge under one of loads and collapse.

OTHER TESTS : Maier Leibnitz carried out a further set of illuminating tests. He took three beams as shown in Fig 1.4, 4.8 m long of the same uniform section. In first case, when supports were in the same level, W reached 13.1 Tons. In second the central suppor



$W_c = 13.1$ Ton

$\sigma_w = 1.2$ ton/cm^2 produced by raising

$W_c = 13$ Ton

Same distance as before - lowered

$W_c = 13.45$ Ton

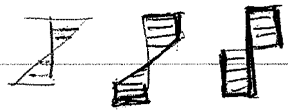
Fig. 1.4.

raised, before the application of any external load, until a bending stress had been developed equal to the usual working stress of $\sigma_w = 1.2$ Ton/cm^2. According to elastic design theory therefore it would not be permissible to apply further external working load. In fact this beam supported almost the same load as the first one. It did not collapse until $W = 13$ Tons. In the third case the central support was lowered by the same distance as it had been raised in the second test. The external loads were then applied and collapse appeared with $W = 13.45$ Ton. These three tests indicated that plastic hinges had once again developed and that the real strength of a continuous beam is unaffected by relative sinking of the supports.

It seemed possible that many of the difficulties inherent in the elastic design of redundant continuous structures, such as building frames would disappear if their behaviour in the plastic range was similar to that of the continnous beams.

prof. Mair - Leibnitz expressed hope and confidence on the possibility of basing designs on plastic behaviour at 1936, but it took 15 years or more before his ideas have been appriciated.

Ealier Work:

G. V. Kazinczy 1914  See Bleich

A. E. H. Love 1892 } recommended the necesity of going beyond elastic limits.

Ewing 1899

Question: Given the loads and general dimensions of a structure, what strength should the members have to produce an optimum solution?

Answer: Assuming elastic - perfectly plastic behaviour the answer can be given. In this respect, the plastic methods provide a major advance over the design efficiency posible with conventional elastic methods. This is so because the plastic method enables the design to be based on a definite factor of safty with respect to a real failure of the structure.

On the contrary, except for determinate structures, the elastic designer has no knowledge of the real failure load, and therefore, of the corresponding factor of safty.

From: CH Massonnet et al., Plasticity in Structural Engineering Fundamentals and Applications, Springer Verlag, Wien, 1979.

1.2 BEHAVIOUR OF AN IDEALISED STRUCTURE

The stress - strain relationship for specimens of structural materials, such as mild steel, has the typical form as shown in F. 1.5. It is almost exactly linear in the "elastic range" until the "upper yield" stress is reached at a.

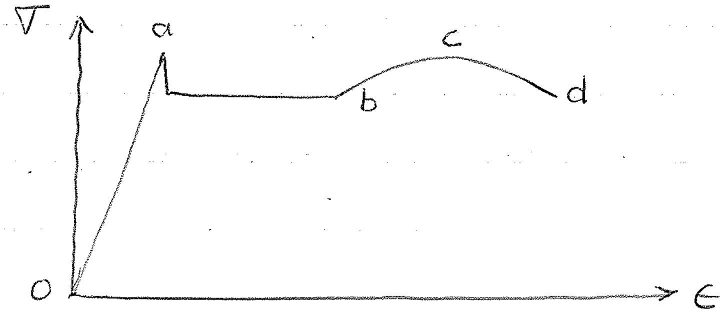


Fig. 1.5 Stress-Strain diagram for mild steel

The stress then drops abruptly to the "lower yield" stress and the strain then increases at constant stress up to the point b, this behaviour being termed "purely plastic" flow. Beyond point b, a further increase of stress is required to produce an increase in strain, and the material is said to be in the "strain-hardening" range. Eventually a maximum stress is reached at c, beyond which increases in strain occurs with decrease of stress until rupture occurs at d. The slope of the linear part oa gives Young's Modulus and the ratio of upper yield stress to the lower is in the order of 1.25 for mild steel.

In analysing structures, various idealizations are made to render the approach manageable. Although the upper yield phenomenon is a real one, it vanishes on cold-working and is often not exhibited by the material of rolled steel sections. Thus one can ignore the upper yield.

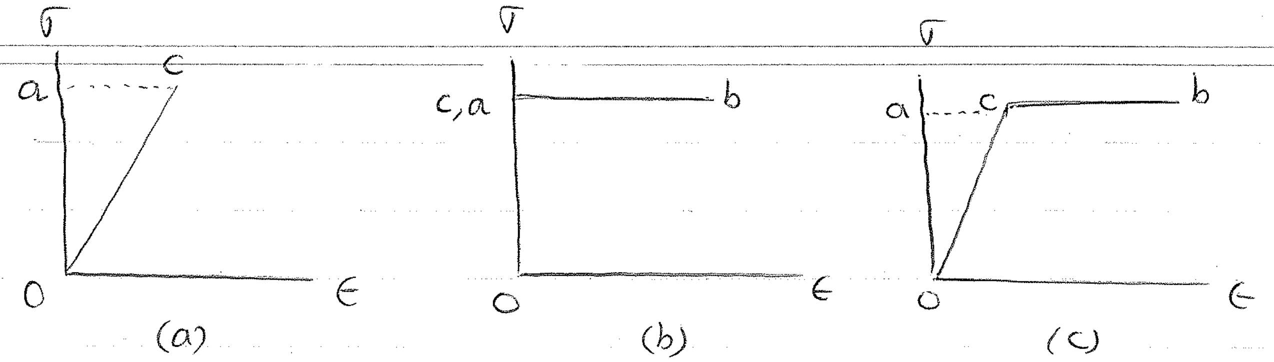


Fig. 1.6 Idealised stress-strain diagrams
(a) elastic
(b) rigid-plastic
(c) elastic-plastic

The effect of strain hardening on the carrying capacity is considerable, especially with small structures. For tall structures, however, the effect of instability is far greater than the strain hardening. In these structures it is reasonable to neglect this effect.

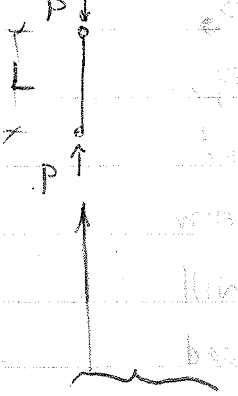
Apart from the above idealisations which are nearly universal, various analytical theories require further simplification. The "elastic" theory, for instance assumes that the safety factor used, safeguards against yielding and thus restricts itself to the portion Oc of $\sigma-\epsilon$ diagram, Fig. 1. (a). The "rigid plastic" theory on the otherhand, is based on the idealisation that $\sigma-\epsilon$ relation Ocb of Fig. 1 (b) reproduces closely the ~~stress~~ - strain for the material concerned. This shows no strain up to point c followed by a pure plastic flow cb. Finally the elastic-plastic theory assumes that the $\sigma-\epsilon$ relation is that given by Ocb in Fig. 1. (c). This is an improvement on the plastic-theory since the latter neglects the elastic strain energy represented by area Oac.

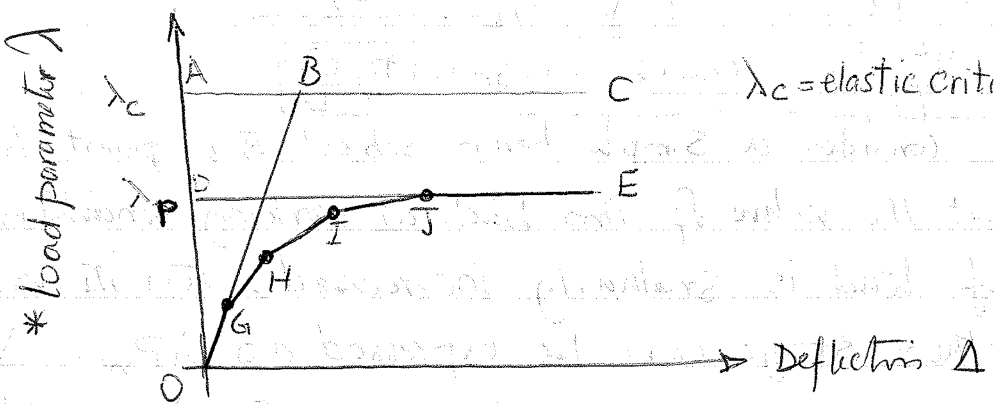
1.3 VARIATION OF GENERALISED DEFORMATION OF STRUCTURES WITH THE LOAD PARAMETER

The calculated values of the lateral deformations of the members of a rigid structure measured orthogonally to their original directions are determined by assumptions made on

1) $\sigma - \epsilon$ for the material,
2) manner in which the external loads are applied,
3) the force to be considered significant for writing Equil. EQs.

$P_{cr} = \frac{\pi^2 EI}{L^2}$



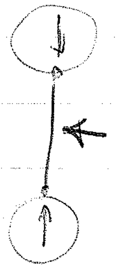


λ_c = elastic critical load factor

Fig. 1.7 Linearised load deformation of structures

Case 1 OAC material is perfectly elastic
applied loads are axial
pin-ended strut axially loaded is an example
λ_c is elastic critical load factor
Equivalent to Euler load of a single member

Case 2 OB The same structure with members transversally loaded. Bending action is dominant, effect of axial load neglected. Typical Linear elastic analysis. G can be obtained by any method such as slope-deflection method.



Case 3 ODE   Simple plastic analysis

rigid-plastic $\sigma-\epsilon$ relationship

neglecting the axial loads

collapse take place suddenly and deformation increase indefinitely.

Apart from neglecting any elastic deformation of the structure prior to collapse, the simple plastic theory involves other assumptions. For example it assumes that an increment of the bending moment at a section always causes an increment of curvature of the same sign and that the magnitude of the curvature always tends to become abruptly large when the bending moment reaches the "fully plastic" moment of the cross section. The theory also assumes that whenever the fully plastic moment is attained at any cross-section, a "plastic hinge" is formed there which can undergo rotation of any magnitude so long as the bending moment remains constant at the full plastic value. Once a hinge is introduced at a section, it is assumed that it will continue to rotate in the same direction.

Case 4 OGHIJ   elastic-plastic material.

at G one plastic hinge is formed in a section of highest bending moment. Then deflection increases at a faster rate (GH) until second hinge is formed. The process is continued until applied load factor reaches the same as rigid plastic.

Assumption are the same as rigid plastic except pre-collapse deflections are not neglected

~~when the effect of axial load is included~~

The non-linear load deformation of structure becomes as shown in Fig. 1.8 which will be discussed later.

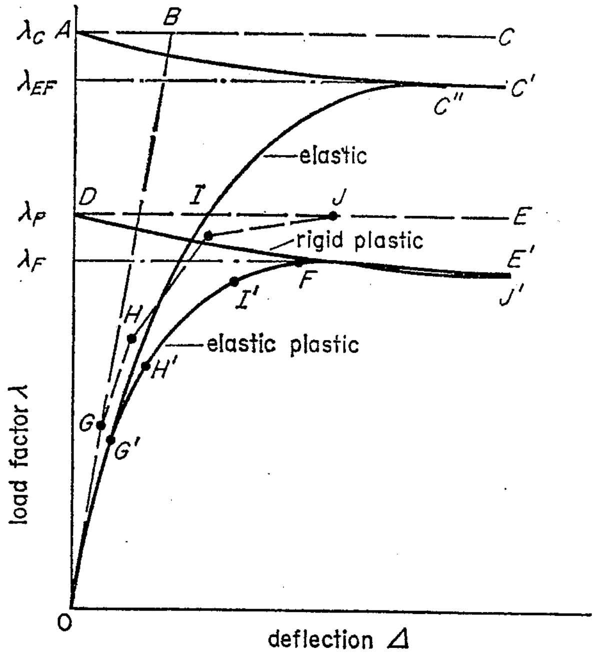


Fig. 1.8

λ_C elastic critical load factor.

OAC elastic, axially loaded, neglecting axial effect

AC' drooping curve due to axial effect – λ_{EF} should be used

OC'' elastic non axially loaded, taking into account the axial effect

λ_P plastic load factor

ODE rigid plastic, neglecting axial effect

DE' rigid plastic including axial effect – λ_F should be used

OGHIJ Elastic-plastic neglecting the axial effect

OG'H'I'J' Elastic-plastic including the axial effect.

Axial load effects are briefly discussed in the next two pages.

The effect of axial load

For small deflection theory, the effect of axial loads in the members can be neglected when deriving the equilibrium equations for a structure. Consider a general member as shown

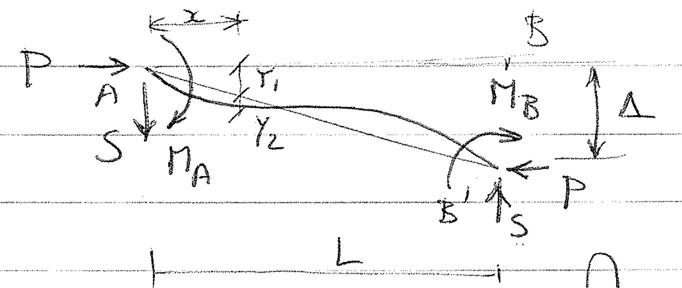


Fig.1.86

Fig.1.86 General deformation of a member

Equilibrium Eq. can be written as

$$M_A + M_B = SL$$

However, when Δ is large or P is large, or both are large, then we should write

$$M_A + M_B + P\Delta = SL$$

Non-linearity in structure is due to $P\Delta$. At a section with distance x, the axial load causes an additional bending moment of manitude $P(y_1 + Y_2)$. PY_1 is due to translation of joint B to B' (i.e relative displacement) and $y_1 = \frac{\Delta x}{L}$, while py_2 is due to actual curvature in the member resulting from the presence of end moments M_A and M_B.

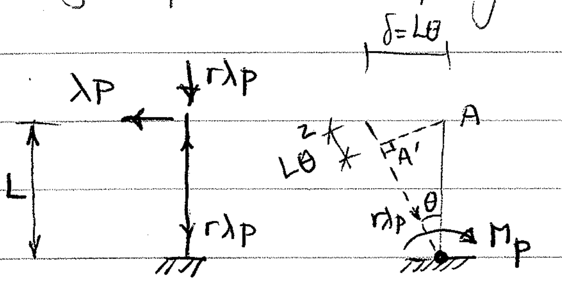
For ideally perfect axially loaded member, until critical load λ_c, axial load does not effect. AT this load, once the structure begins to deform, the axial load starts to play a part in the equilibrium of the members, resulting in modifying the straight line AC to drooping curve AC'. That is to say, to maintain equilibrium, as deflection increase, the externally applied load have to be reduced.

Similarly, for non-axially loaded elastic structure, taking into account the effect of axial load, modifies the lateral displacement in a non-linear manner as indicated by OC".

For rigid-plastic behaviour DE changes to DE' when the axial load is considered. For more important the consideration of axial load undermines the basic fondamentals of the rigid plastic Theory. i.e. axial load violates the three requirements of equilibrium, Yield and Mechanism. e.g. for equilibrium virtual work equation will contain second order terms. Yield gets violated as well. e.g. bending moment is not any more necessarily highest under point loads and joints. The worst effect is to make collapse to take place before the formation of a mechanism and at a load factor λ_F which is below the pridicted λ_p.

Finally, the axial load may alter the mode of deformation completely in such a way That the sequence of hinge formation would be entirely different from those pridicted by the piece-wise linear elastic-plastic Theory.

Rigid plastic drooping curve:



The work equation for post collapse condition becomes

$$M_p\theta = (\lambda P)(L\theta) + (r\lambda P)(L\theta^2)$$

substitution of $\delta = L\theta$ yields

$$\delta = (M_p - \lambda PL)/r\lambda P$$

which The equation of the drooping curve.

This idea can be generalized from a member to a frame.

Notice That δ-P curve of The above equation is non-linear.

1.4 THE COLLAPSE OF BEAMS

An engineering structure has to satisfy many functional requirements

1. Strong enough to resist the external loading
2. Stiff enough not to deflect unduly

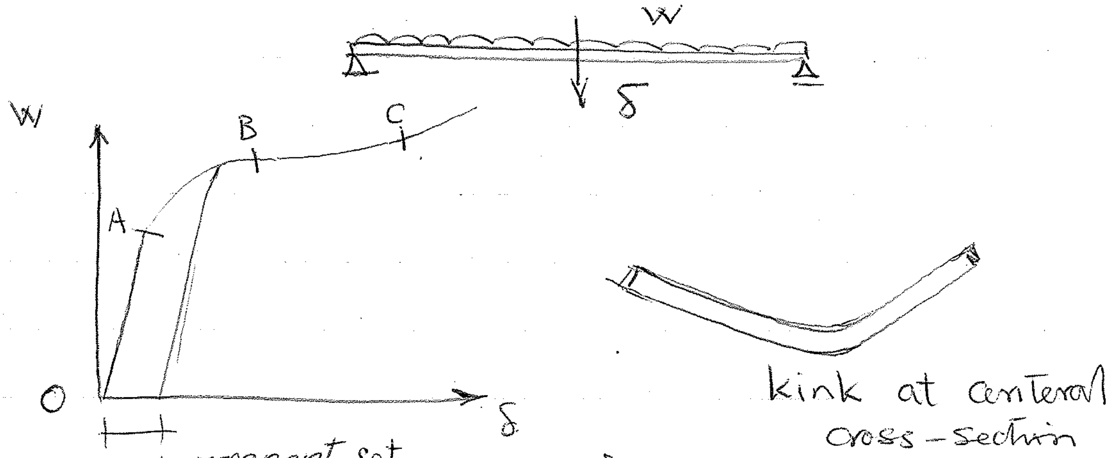


Fig. 1.9

OA elastic and deflections are fully recoverable on removal of load

AB In this part the rate of deflection is higher and with removal of load a kink, as shown will stay.

BC Further increase of W, leads to a very rapid deflection, with slight raising due to strain hardening. The slope depends on material and loading condition. The deflection is so much that Beam reaches to the limit of its usefulness

An approx. curve can be considered as Fig. 1.10

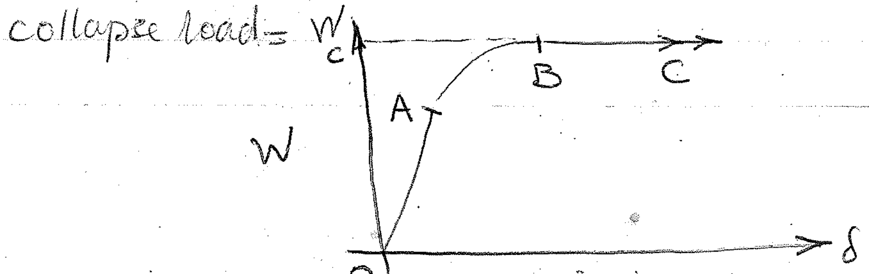


Fig. 1.10

~~In this curve deflections increase without limit, at a~~ constant W_C along portion BC. The load W_C is called a <u>collapse load</u> of the beam. Since W_C is constant, then the bending moments in the collapsing beam are also constant. $(M = \frac{W_C L}{8})$ The unrestricted deflection at collapse are produced by extremely localized deformation at the centeral "kink" in the beam. This centeral cross-section behaves like a "<u>rusty</u>" hinge connecting together the two halves of the beam.

In its collapse state, the beam has shape as Fig. 1.9, and almost all the large deflection of Fig. 1.9 may be considered as due to the rotation of the central hinge. This hinge is known as a <u>plastic hinge</u>, and <u>it occurs at the section of greatest bending moment in the beam.</u>

For this simple model, the strength of the structure is given by its collapse load W_C, and is not immediately related to the elastic behaviour, as is conventionally assumed. Although the load at which yield occurs (point A) bears some more or less definite ratio to the collapse load for simply supported beam, this ratio varies for different types of structures.

However, the second structural requirement, that of <u>stiffness</u>, must be considered. For a large class of building structure like multi-storey steel frames designer is seldom worried by deflexions

~~That is, if the members are proportioned on the~~ basis of their strengths, and then calculations are made to estimate deflexions, these second calculations are rarely critical. It is for this class of structures that plastic theory has been developed.

1.4.1 THE FULL PLASTIC MOMENT

The load deflection curve of Fig. 1.9 (or the idealized curve of Fig. 1.10) results from a certain moment – curvature relationship for the cross-section of the simply supported steel beam. For this simple example M – K and its idealised form is as Figs. 1.11 and 1.12, respectively.

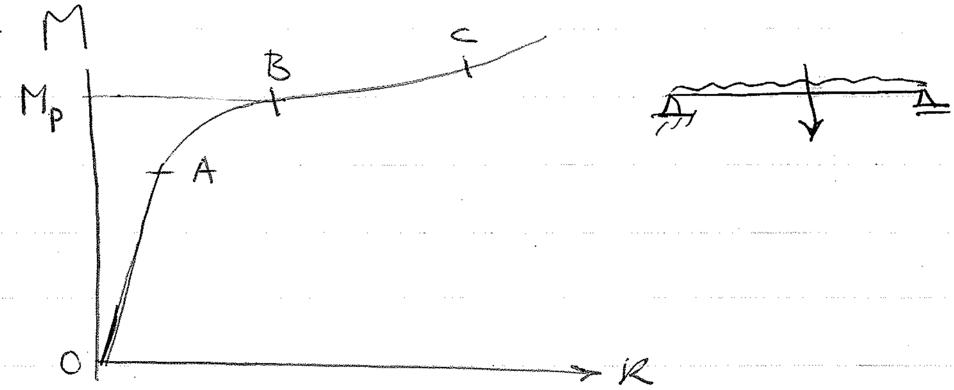


Fig. 1.11

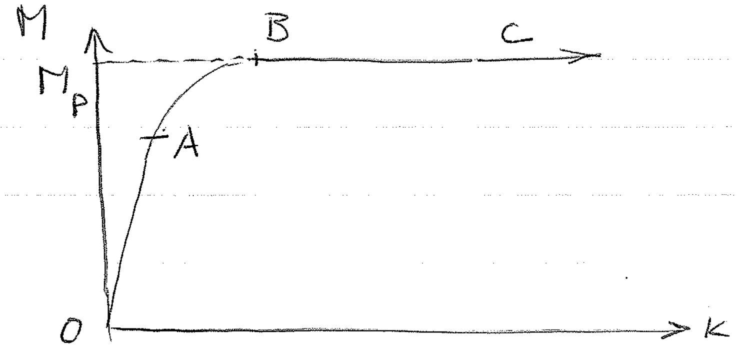


Fig. 1.12

As before, in an ideal form, the curvature increases indefinitely at a constant value of the bending moment, M_p. Evidently the value of M_p corresponds to the moment acting at the plastic hinge. This moment is known as the full plastic moment.

In this curve

OA	elastic and reversible behaviour, a small increase in bending moment produces a small proportional change in curvature
AB	From A yield will take place with a corresponding greater increase in curvature
BC	At B plastic hinge is fully developed, and unlimited increase in curvature, or rotation, can occur at the constant bending moment M_p.

IN SIMPLE PLASTIC THEORY a knowledge of the value of the full plastic moment (FPM) is all that is required. When F.P.Ms are known for members of structure, then its collapse load can easily be found, even if the frame is very complex.

Similarly, the design of a frame to carry given loads consists in the assignment of certain minimum values of FPM to the members. A simple bend test producing a load-deflexion curve will give an estimate of the value of full plastic moment, thus furnish the designer with the information he needs.

However, it is natural to try to relate the full plastic behaviour in bending to the observed stress-strain relationship in simple tension. Then designer can tabulate or calculate sets of plastic moduli which, when multiplied by yield stress of the material, would give the required values of the full plastic moments.

$$Z_p = \frac{M_p}{\sigma_0} \Rightarrow M_p = Z_p \sigma_0$$

~~1.4.2 THE BENDING OF BEAMS~~

Consider $\sigma - \epsilon$ curve for a mild steel and idealise it as shown.

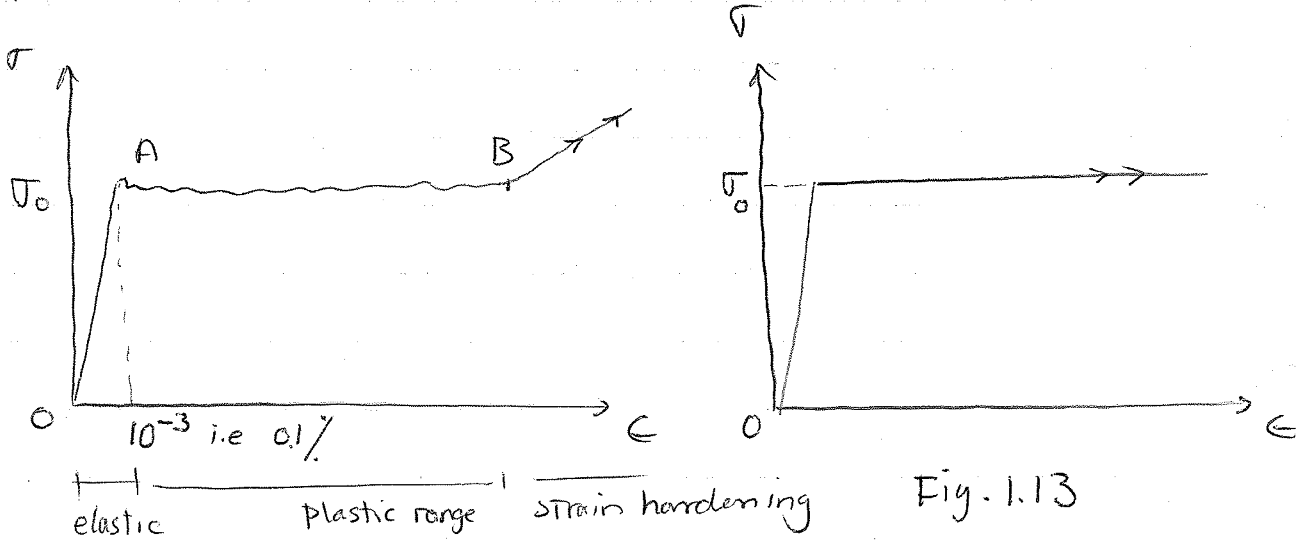


Fig. 1.13

Strain at B is at least 10 times as much as strain in A. laying in 2%. Thus plastic range is sufficiently long to extend it and ignore strain hardening.

Assumptions :

a) the beam is bent by pure bending, and so the effects of axial and shear forces are not included

b) plane cross section remains plane

c) the deformations are small, thus only normal longitudnal stresses are considered, σ_x.

d) the relation between σ_x and ϵ_x is in flexture the same as compression or tension.

e) The above idealised $\sigma - \epsilon$ is assumed.

1.4.3 VARIOUS STAGES of BENDING OF A BEAM.

Rectangular cross section of $b \times 2d$ is considered.
At the beginning of yield, the yield moment is given as

$$M_y = \frac{1}{2} b d \sigma_0 \left(\frac{4}{3} d\right) = \frac{2}{3} b d^2 \sigma_0$$

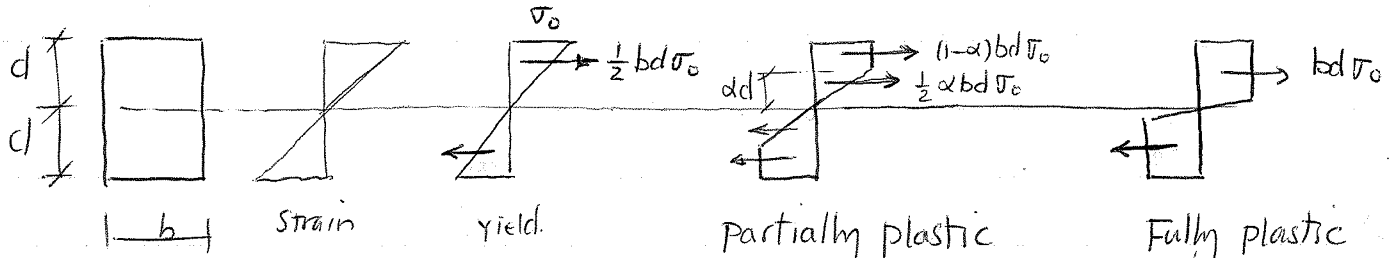


Fig. 1.14

elastic modulus $Z_e = \frac{M_Y}{\sigma_0} = \frac{2}{3} b d^2$

In partially plastic state

$$M_R = (1-\alpha) b d \sigma_0 \left[(1+\alpha) d\right] + \frac{2}{3} \alpha^2 b d^2 \sigma_0$$

$$M_R = b d^2 \sigma_0 \left(1 - \frac{1}{3}\alpha^2\right)$$

As a check put $\alpha = 1$, then $M_R = M_y$.

As more and more cross-section passes from elastic to plastic range ($\alpha \longrightarrow 0$) M_R approches its limiting value

$$M_R\big]_{\alpha=0} = M_P = b d^2 \sigma_0 \qquad M_R = M_P\left(1 - \frac{1}{3}\alpha^2\right)$$

plastic modulus is defined analogously as

$$Z_P = \frac{M_P}{\sigma_0} = b d^2$$

shape factor is defined as $\nu = \frac{Z_P}{Z_e}$
which is 1.5 for rectangular sections. It is $\doteqdot$ 1.15 for I sections

~~1.4.4 SHAPE OF THE PLASTIC ZONE~~

For partially plastic section we had

$$M_R = bd^2\sigma_0\left(1-\tfrac{1}{3}\alpha^2\right)$$

For fully plastic section

$$M_p = bd^2\sigma_0$$

Hence

$$M_R = M_p\left(1-\tfrac{1}{3}\alpha^2\right)$$

This equation can be used to determine the shape of the plastic zones for simple beam of rectangular cross section

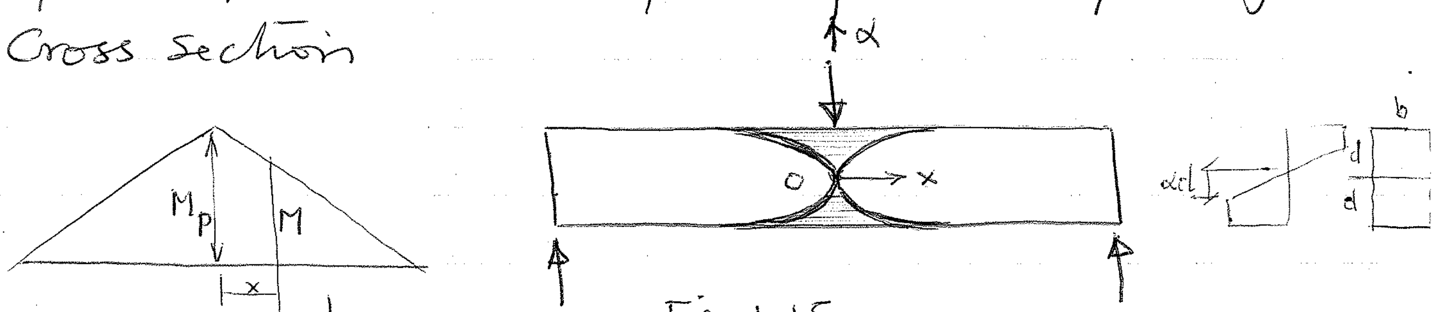


Fig. 1.15

In above example suppose the load is just enough to cause the full plastic moment to be developed. At any section x of the beam bending moment is given by

$$M/M_p = \frac{l-x}{L} = 1-\frac{x}{L}$$

i.e.

$$M = M_p\left(1-\frac{x}{L}\right)$$

Equating M with M_R

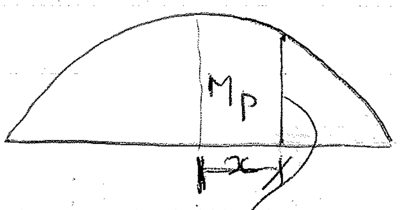
$$M_p\left(1-\frac{x}{L}\right) = M_p\left(1-\tfrac{1}{3}\alpha^2\right)$$

We have

$$\alpha = \sqrt{\frac{3x}{L}}$$

The plastic zones under the central point load meet at this point, and fall away sharply; the plastic hinge is confind to an infinitesimal length.

For a uniformally distributed loading the plastic zone is obtained as



$$M = M_p\left(1-\frac{x^2}{L^2}\right)$$

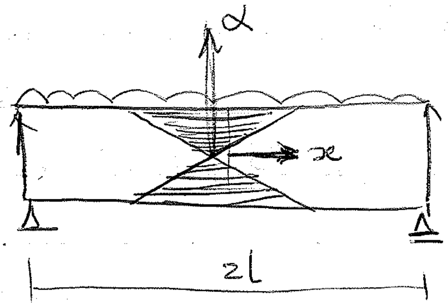


Fig. 1.16

$$\therefore \quad 1-\frac{1}{3}\alpha^2 = \left(1-\frac{x^2}{L^2}\right) \quad \text{and} \quad \alpha = \sqrt{3}\,\frac{x}{L}$$

1.4.5 Moment – curvature of a cross-section

Corresponding to the general expression for M_R in the partially plastic state, the curvature of the section can be calculated in terms of α

$$(1) \qquad \boxed{M_R = M_p\left(1-\frac{1}{3}\alpha^2\right)}$$

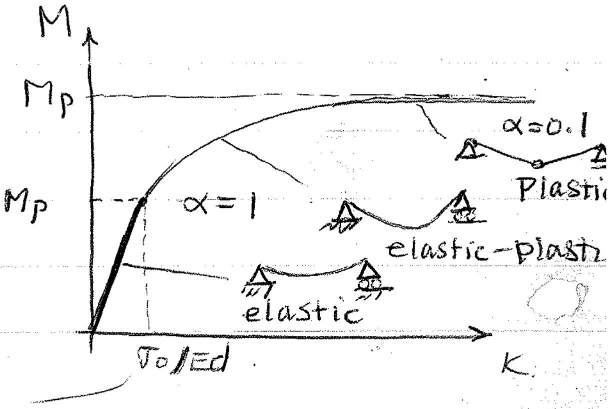
The usual elastic formulae must hold for the central elastic **core** of depth $2\alpha d$. So the curvature can be written as

$*$ $\quad \varepsilon = K y$ where $y = \alpha d$ and $\varepsilon = \varepsilon_0 = \frac{\sigma_0}{E}$

$$\therefore K = \frac{\varepsilon}{y} \qquad (2) \Rightarrow \boxed{K = \frac{1}{\alpha}\cdot\frac{\sigma_0}{Ed}} \qquad 1 > \alpha > 0$$

The above (1)&(2) equations are parametric expressions for M–K curve in elastic–plastic range.

from (1) ←



from (2)

Fig. 1.17 M–K curve

1.4.6 I – Sections (shape factor)

A similar analysis is made for I-section. A universal Beam has 3° taperd flanges and a typical profile as Shown in Fig. 1.18, replaced by rectangular flanges of uniform thickness

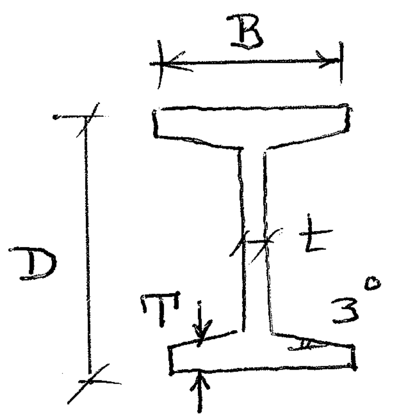


Fig. 1.18

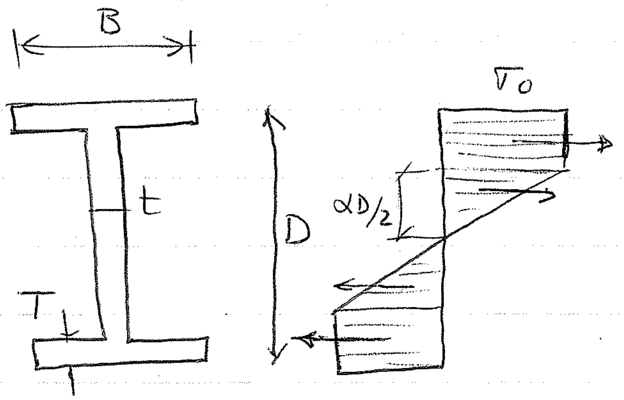


Fig. 1.19

The moment of resistance in partially plastic state is

$$M_R = \left[BT(D-T) + \left(\frac{D}{2}-T\right)^2 t - \frac{1}{12}D^2 t\alpha^2\right]\sigma_0$$

in this expression $\frac{D-2T}{D} > \alpha > 0$

(Seperate analysis is needed when yield has just started)

As an example for 12×5 Universal Beam 48 kg/m

$B = 125\ mm \quad D = 310\ mm \quad T = 14\ mm \quad t = 8.9\ mm$

$$M_R = (518000 + 177000 - 71000\alpha^2)\sigma_0$$

$$= \underbrace{(695000 - 71000\alpha^2)}_{\text{given in } mm}\sigma_0 = Z_p\sigma_0 \text{ when } \alpha \to 0,\ M_R \to M_p$$

i.e. ∴ $Z_p = 695000$

Z_p when calculated accurately is $705000\ mm^3$

Thus 1% difference, because of replacing with rectangular flanges

Elastic modulus

$$Z_e = \frac{1}{D}\left[\left\{BT(D-T)^2 + \frac{1}{3}BT^3\right\} + \frac{4}{3}\left(\frac{D}{2}-T\right)^3 t\right]$$

numerical value for the same section

$$Z_e = 495000 + 107000 = 602000 \text{ mm}^3$$

compared with 611000 from Table

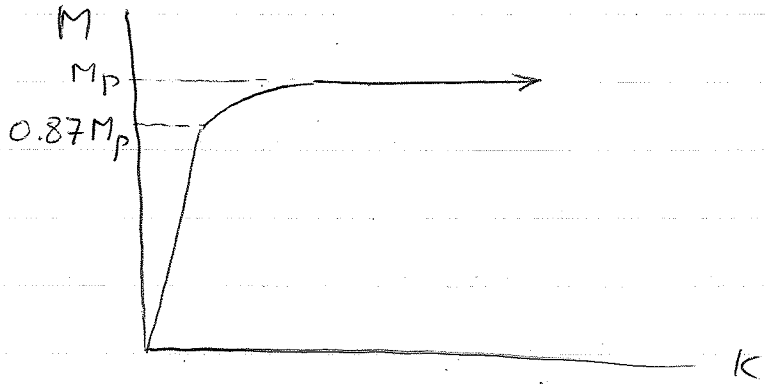
Thus the Shape factor is

$$\nu = \frac{705}{611} = 1.15$$

Thus in bend test of one of these section, yield will not occur until bending moment is $0.87 M_p$. The M–K curve is as

$$\nu = \frac{Z_p}{Z_e} = \frac{M_p}{M_y} = 1.15$$

$$M_y = \frac{M_p}{1.15} = 0.87 M_p$$



Curvature is obtained $K = \frac{2\sigma_0}{ED\alpha}$ as in rectangular section

Fig. 1.21

Fig. 1.20

It is nice to see M–k for various sections

for rod section

$$Z_p = Z_e = \frac{1}{2}AD$$

i.e. $\nu = 1$

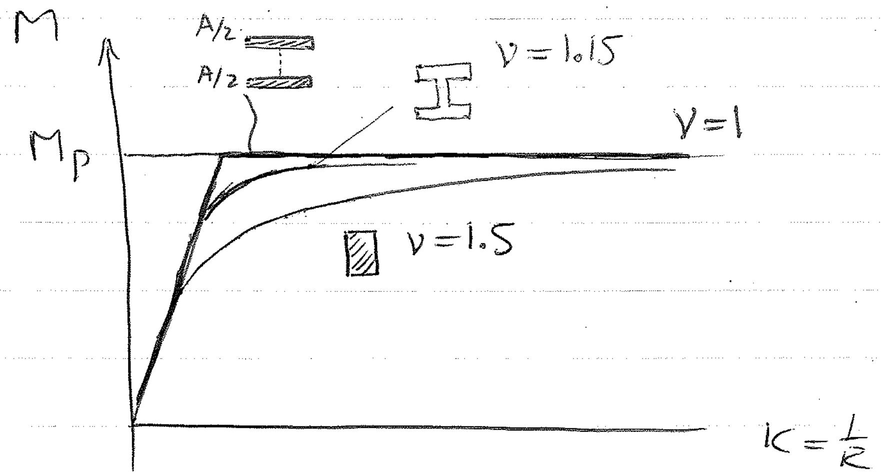


Fig. 1.21

Table plastic moduli Z_p and shape factor ν

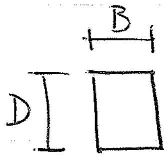
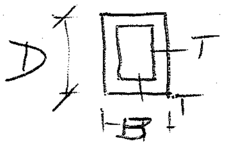
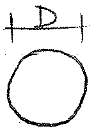
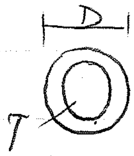
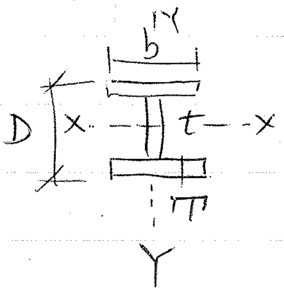
		Z_p	ν
solid rectangular		$\frac{1}{4}BD^2$	1.5
Hollow rectangular		$BT(D-T)+\frac{1}{2}T(D-2T)^2$	$B=D$ $T=0.05D$ $\nu=1.18$
solid circular		$\frac{1}{6}D^3$	$\frac{16}{3\pi}=1.70$
Hollow circular		$\frac{1}{6}D^3\left[1-\left(1-\frac{2T}{D}\right)^3\right]$ $\left\{ T \ll D : TD^2 \right.$	$T=0.05D$ $\nu=1.34$ $\left\{ \begin{array}{l} T \ll D \\ \nu=\frac{4}{\pi}=1.27 \end{array} \right.$
I-Section		Axis XX $BT(D-T)+\frac{1}{4}t(D-2T)^2$ Axis YY $\frac{1}{2}TB^2+\frac{1}{4}(D-2T)t^2$	# 1.14 Universal beams # 1.60 universal beams

Fig. 1.22

1.47 Z_p FOR SECTION WITH ONE AXIS OF SYMMETRY:

Up to now zero stress axis concided with the usual neutral axis of elastic theory of bending. This is true only for sections with two axes of symmetry.

A general cross section having one axis of sym. is shown in Fig. 1.23 Full plastic stress distribution is

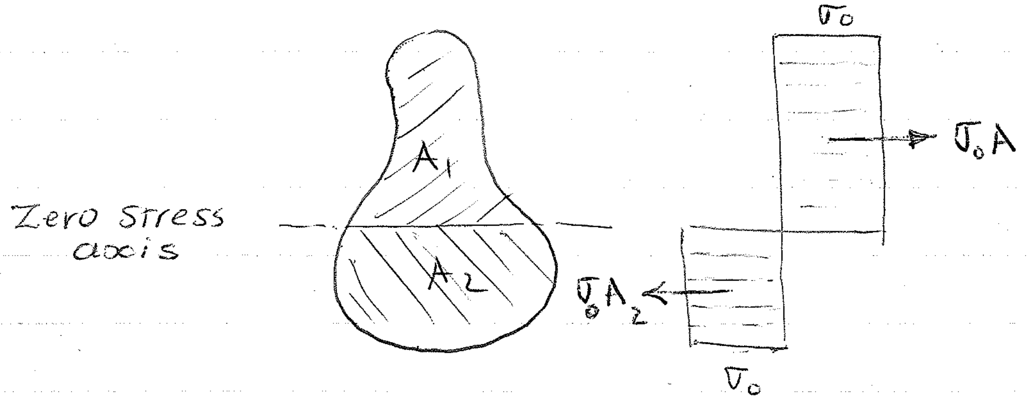


Fig. 1.23

also shown. The total force on top half is $\sigma_0 A_1$ and on bottom half $\sigma_0 A_2$. In the absence of axial load, $\sigma_0 A_1 = \sigma_0 A_2$ i.e $A_1 = A_2$

Thus infact

Zero stress axis = An equal area axis

In general this does not pass through the center of gravity of section as in elastic neutral axis.

Writing a general equation for elastic-plastic bending, it is possible to trace the gradual shift of zero-stress axis as the bending moment is increased, from centroidal position to equal area position of full plasticity.

EXAMPLE : T section of equal rectangular plates

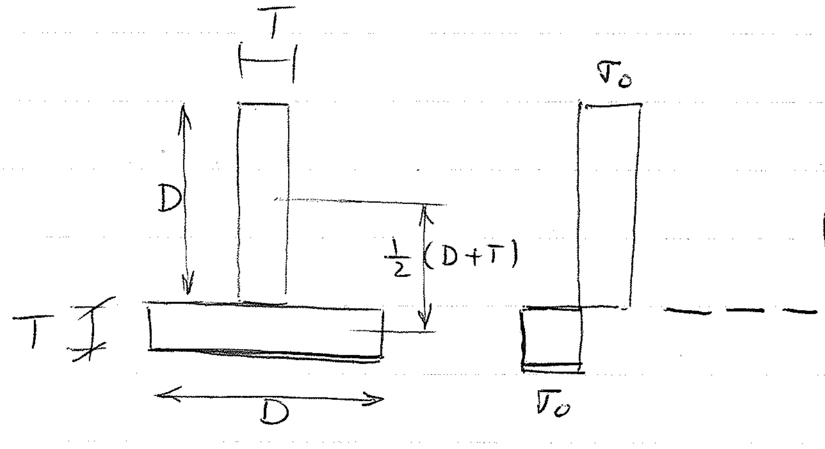


Fig. 1.24

Zero stress axis = equal area axis

The force on each rectangular is $\sigma_o DT$ and the arm is $\frac{1}{2}(D+T)$.

$$Z_p = \frac{1}{2}DT(D+T)$$

Compared with

$$Z_e = \left(\frac{DT}{3D+T}\right)\left[\frac{1}{3}(D^2+T^2)+\frac{1}{2}(D+T)^2\right]$$

Thus for $D/T = 8$ say, the shape factor is

$$\nu = Z_P/Z_e = \frac{675}{373} = 1.81.$$

1.47 DESIGN WITH COVER PLATES

plastic moduli = first moment of area about the zero stress axis

Now if a plate is added as shown in Fig. 1 , with area "a" then zero stress axis should be lowered by $\frac{a}{2}$

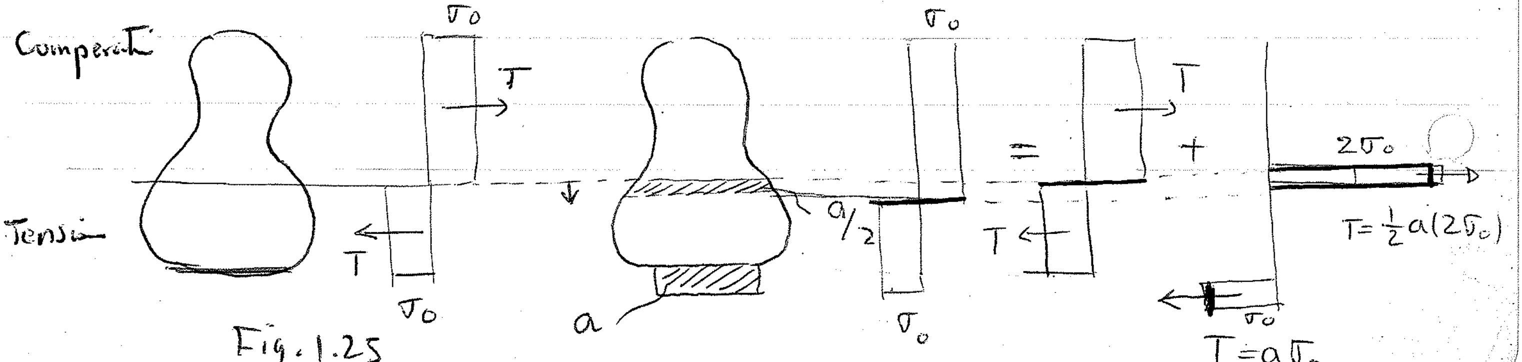


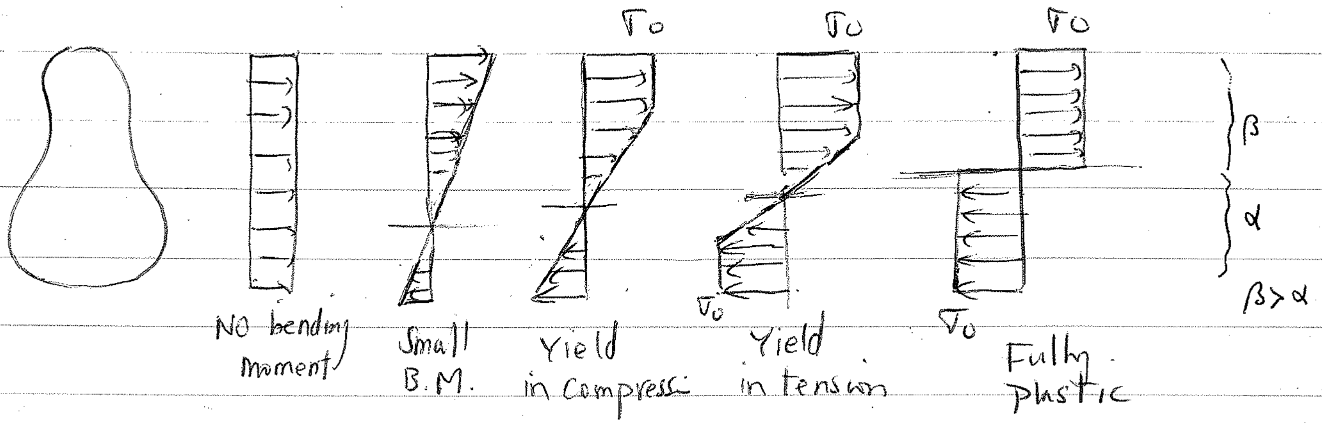
Fig. 1.25

Final fully plastic stress distribution may be thought of two parts:

The original full plastic moment, is increased by a moment due to the flange plate force $a\sigma_0$, and a force due to the fictitious stress $2\sigma_0$ acting on the area $a/2$, that has been transferred from tension to compression.

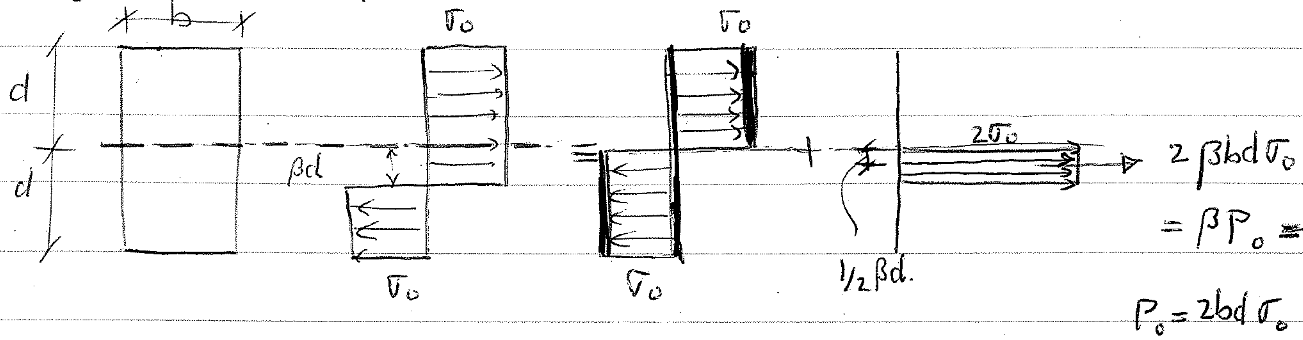
1.5 EFFECT OF AXIAL LOAD ON M_p

If an axial load is applied at a cross section of a short column, then the load give rise to a uniform compressive load over the section. Addition of a small moment will produce linear variation of elastic stress. Further increase on bending moment, with the axial load remaining constant, will eventually cause yield on one face then followed by other face. During this process zero-stress axis will shift as shown towards the final position in the fully plastic state.



1.5.1 RECTANGULAR SECTIONS Fig. 1.26

It is this full plastic state that is of interest. To see the effect of axial load on the full plastic moment, a rectangular cross section is considered.



$P_0 = 2bd\sigma_0$
squash load

Fig. 1.27

An axial load P is supposed to act, in addition to a bending moment M_p sufficient to cause the whole section fully plastic. In order to have a net resultant force P, the zero-stress axis must shift from the centre line, by an amount say βd.

Considering the distribution as two part (see Fig. 1.27)

$$P = 2\beta b d \sigma_0 = \beta P_0 \qquad \text{where} \quad P_0 = 2bd\sigma_0$$

$$M_p = M_{p_0} - P\left(\tfrac{1}{2}\beta d\right) = (1-\beta^2) M_{p_0} \qquad \text{where } M_{p_0} = bd^2\sigma_0$$

Where P_0 is the 'squash load' of section in the absence of bending moment, and M_{p_0} is the value of full plastic moment in the absence of axial load.

The above equations can be combined as

$$\left(\frac{M_p}{M_{p_0}}\right) + \left(\frac{P}{P_0}\right)^2 = 1 \qquad \text{i.e. eliminate } \beta$$

which give the characteristic of the following curve.

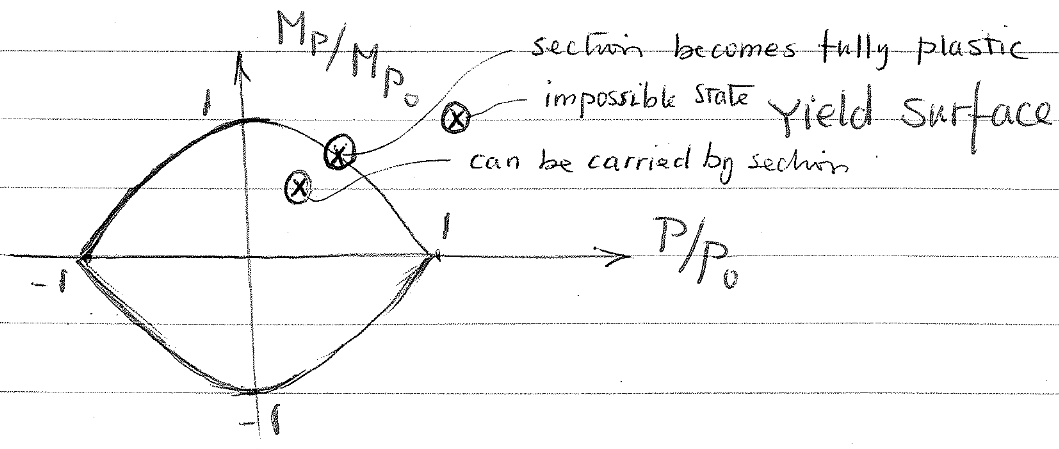


Fig. 1.28 Effect of axial load on M_p

P_0 = squash load = $2bd\sigma_0$

M_{p_0} = plastic moment in the absence of P, i.e. $M_{p_0} = bd^2\sigma_0$

Fig. 1.28 is an important concept in plastic Theory, and represents a yield surface, which is doubly symmetric. A point in the plane of the figure represents, for a given rectangular section having known values of M_{P_0} and P_0, a certain combination of bending moment and axial load. If the point lies within the convex boundary, then the combination is one that can be carried by the cross section. A point on the boundary of the convex yield surface represents a combination of bending moment and axial load that just cause the section to become fully plastic. A point outside the convex region represents an impossible state.

1.5.2 EFFECT OF AXIAL LOAD ON I-SECTIONS

Similar analysis for I-section leads to

$$M_P = M_{P_0} - \beta^2 t D^2 \sigma_0$$

$$P = 2\beta t D \sigma_0$$

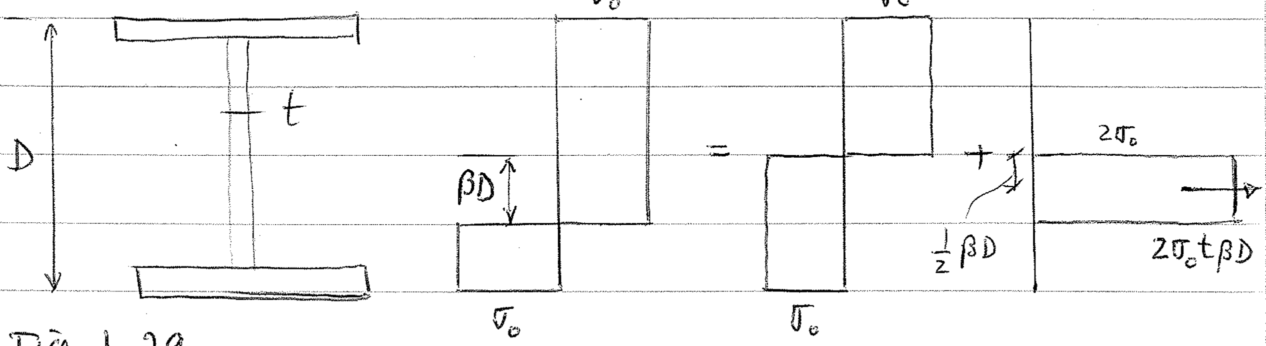


Fig. 1.29

If the mean axial stress is denoted by $P = pA$, where A is the total cross sectional area and if $n = p/\sigma_0$ then the above EQs may be combined as

$$Z_p = \frac{M_P}{\sigma_0} = Z_{P_0} - \beta^2 t D^2 = Z_{P_0} - \left(\frac{A^2}{4t}\right) n^2$$

For 12X5 UB 48 kg $A = 6080\ mm^2$ $t = 8.9\ mm$ This EQ. becomes

$$Z_p = 705000 - 1038000n^2 \quad (1)$$

This EQ. hold for small values of axial forces. For Large values of n, The zero-stress axis moves from Web to flange and a new analysis should be made. e.g. for $n > 0.412$ and 12X5 UB 48kg, Z_p is obtained as

$$Z_p = 74480(1-n)(11.66+n) \quad (2)$$

The above Two EQS. are plotted as

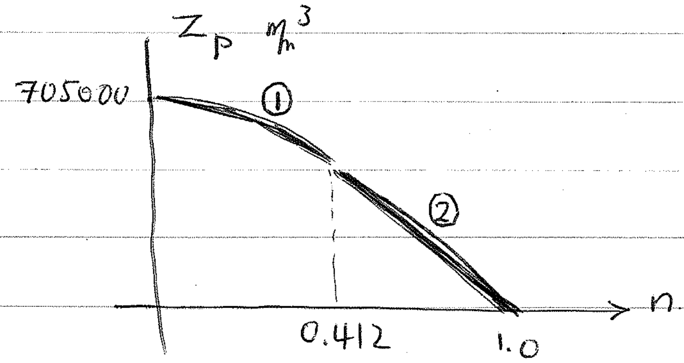


Fig.1.30 quarter of yield surface for I-Section.

1.5.3 GENERAL CASE

For a cross-section with one axis of symmetry, it may be supposed that the axial load P (acting in the plane of the original zero-stress axis) Causes the zero-stress axis to shift so that an area "a" is Transferred from tension to compression (or vice versa)

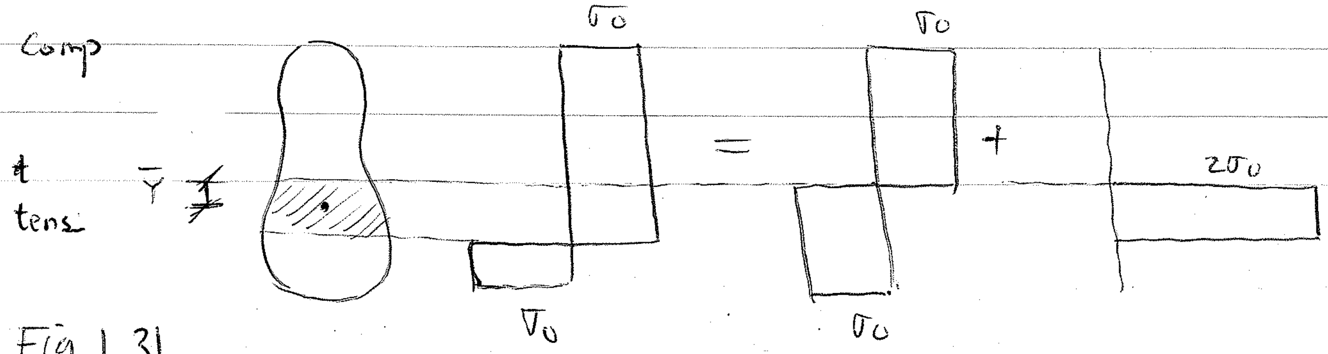


Fig. 1.31

From this figure, if the centre of the transferred area "a" is distant $\bar{y}$ from the original zero-stress axis, then

$$P = 2\sigma_o a$$
$$M_p = M_{p_o} - P\bar{y}$$

That is

$$\boxed{Z_p = Z_{p_o} - 2a\bar{y}}$$

which is a simple general expression for computing the reduction in plastic modulus due to axial load.

* If wanted T section can also be presented

1.6 EFFECT OF SHEAR FORCE ON M_p

The effect of shear force on a beam of general cross-section is more complex than that of axial load. With axial load, the resulting stresses could be superimposed directly on the bending stresses, since they were all longitudinal. Shear combined with bending gives rise to a two-dimensional stress system, which make the problem more involved.

The special case of I-section can be dealt with approximately. We assume uniform distributed shear stresses over the web and ignore the contribution of flanges (as in elastic case). In this case a imperical solution may be obtained.

The full plastic moment of an I-section, previously obtained, is

$$M_p = BT(D-T)\sigma_o + \left(\frac{D}{2}-T\right)^2 t\,\sigma_o$$
$$= M_f + M_{w_o}$$

M_f and M_{w_o} are the contributions of flanges and web.

Suppose now shear force F acts on the web causing a uniform shear stress τ as

$$F = (D-2T)\,t\,\tau$$

If the web remains plastic, then σ (bending) will be reduced below the value σ_o. Using Von Mises

$$\sigma^2 + 3\tau^2 = \sigma_o^2$$

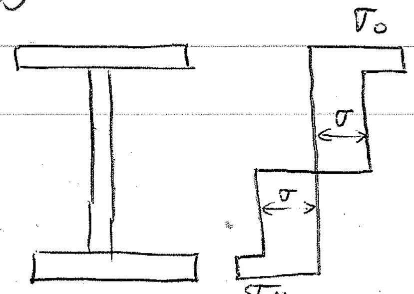


Fig. 1.32

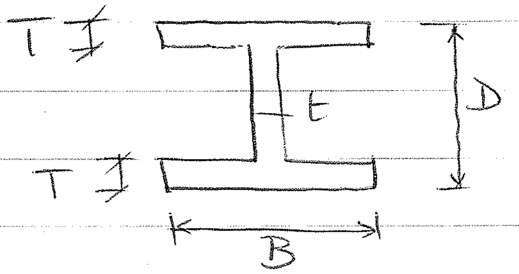
The stress distribution under combined shear and bending sufficient to cause full plasticity will be as shown in Fig. 1.32.

METHOD : The contribution from the flanges remain unaltered, but the contribution from the web is reduced by the ratio σ / σ_0.

EXAMPLE:

D×B (inch)

Consider a section 12×5 UB 48 kg with shear force 300 kN and yield stress $\sigma_0 = 250\ N/mm^2$.



From Table:

$A = 6080\ mm^2$

$t = 8.9\ mm \quad T = 14\ mm$

$B = 125\ mm \quad D = 310\ mm$

Considering uniform shear

$$\tau = \frac{F}{A_W} = \frac{300{,}000}{(282)(8.9)} = 119.4\ N/mm^2$$

From yield criterion $\sigma^2 + 3\tau^2 = \sigma_0^2$ we have

$$\frac{\sigma}{\sigma_0} = \sqrt{1 - 3\left(\frac{\tau}{\sigma_0}\right)^2} = \sqrt{\left\{1 - 3\left(\frac{119.4}{250}\right)^2\right\}} = 0.562$$

The full plastic moment of I-section from previous page can be rewritten as

$$M_P = M_f + M_{w_0} = M_f + \left(\frac{D}{2} - T\right)^2 t\, \sigma_0$$

dividing by σ_0 (definition of plastic modulus)

$$Z_{P_0} = Z_f + (141)^2 (8.9) = Z_f + 177000.$$

According to the above method the contribution of web (177000) should be reduced by σ/σ_0. Therefore

$$Z_p = Z_f + 177000\,\frac{\sigma}{\sigma_0}$$

However, from previous relationship we have

$$Z_f = Z_{p_0} - 177000$$

Therefore

$$Z_p = Z_{p_0} - 177000 + 177000\,\frac{\sigma}{\sigma_0} = Z_{p_0} - 177000\left(1 - \frac{\sigma}{\sigma_0}\right)$$

For $\sigma/\sigma_0 = 0.562$ the above relation yields

$$Z_p = Z_{p_0} - 177000\,(0.438)$$

For $Z_{p_0} = 705000$ we have

$$Z_p = 705000 - 177000\,(0.438) = 627000$$

These results are depicted in the following figure (•).

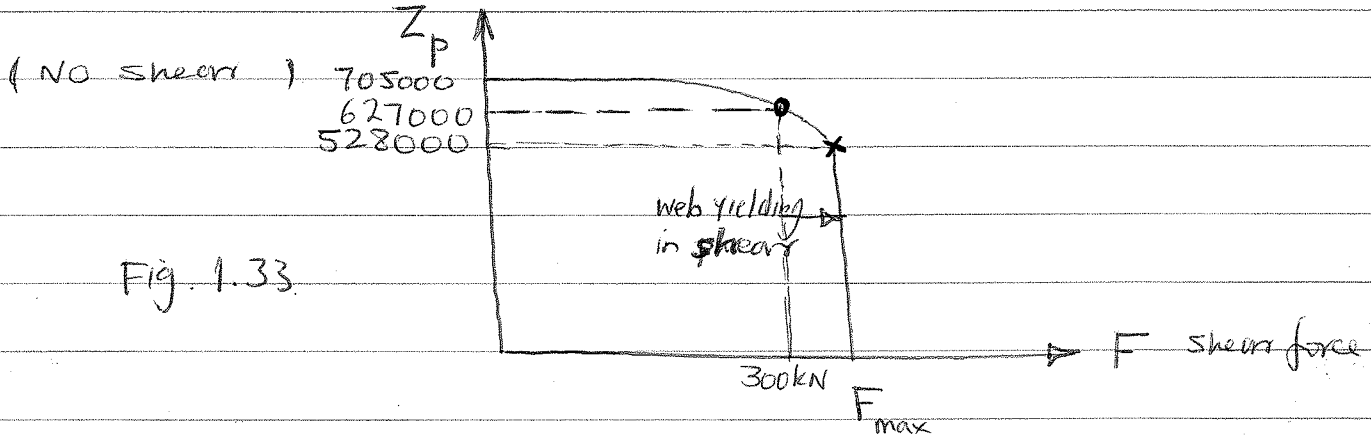


Fig. 1.33

For small shear, very little reduction in Z_p. When the whole of web is yielding in shear at an average stress $\frac{250}{\sqrt{3}} = 144\ N/mm^2$, the flanges remain to contribute a plastic modulus $Z_f = 528000\ mm^3$ (x).

NOTE: When both axial force and shear force are present, then there will be the danger of local buckling and thus restrictions should be imposed.

1.7. THE LOAD FACTOR

A structure is always designed with some margin of safty ; that is Working loads < Collapse loads.

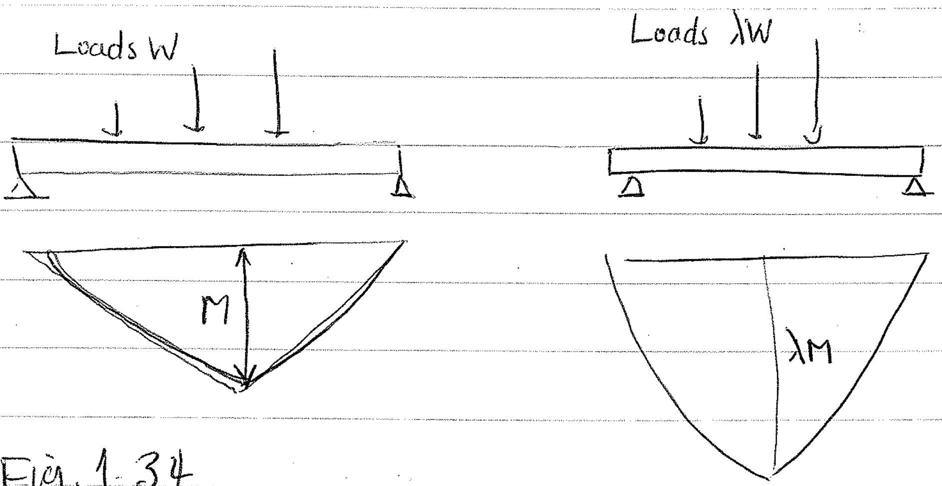


Fig. 1.34

Suppose a simple beam is subjected to a set of working loads and max bending moment is M. If all loads are multiplied by λ, Then the max bending moment will be λM. Collapse will occur when maximum bending moment $\lambda_c M$ reaches the full plastic value. The value λ_c is called **the collapse load factor,** that is, it is the ratio of the collapse load to the working load on the structure.

The primary function of a load factor is to ensure that a structure will be safe under service conditions. However, the load factor performs several duties, since margins of safty are required to cover uncertainities in the values of loads, imperfections etc. ...

DESIGN RECOMMENDATION BS449

According to BS449 the plastic designer would carry out a design under

Dead + Superimposed load $\lambda_c = 1.75$

Dead + Superimposed load + Wind load $\lambda_c = 1.40$

Naturally the more severe of these two would govern the final design.

These values for particular structures and codes may be increased or decreased depending on the designer's estimate of uncertainty of the design conditions.

1.8 APPLICATIONS TO OTHER MATERIALS

plastic method can be applied to frames of any materials provided the members behave closely in accordance with the plastic hinge assumptions, i.e. whenever the bending moment reaches a critical value, a plastic hinge forms and can undergo extensive rotation while the bending moment remains sensibly constant.

Reinforced concrete members often exhibit a limiting moment at which quite large hinge rotation develops but eventually the moment falls off with further increase of rotation. The applicability of the plastic methods to reinforced concrete frame, therefore depends crucially on the rotation capacity and ductility of the hinges.

Baker (1970) has described an "ultimate load" method for designing reinforced concrete frames, which differs from the simple plastic methods in that it requires only limited rotations at the plastic hinges. The rotation capacity of hinges in reinforced concrete beams is studeed by Cranton and Reynolds (1970) showing in certain circumstances a good measure of ductility is available. It is also shown that:

For rectangular portal frames, the simple plastic theory can provide close estimates of actual collapse loads.